Point-Set Topological Spatial Relations*

Max J. Egenhofer

National Center for Geographic Information and Analysis and Department of Surveying Engineering, 107 Boardman Hall, University of Maine, Orono, Maine 04469, U.S.A.

and Robert D. Franzosa

Department of Mathematics, 417 Neville Hall, University of Maine, Orono, Maine 04469, U.S.A.

Abstract

Practical needs in the realm of Geographic Information Systems (GISs) have driven the efforts to investigate formal and sound methods to describe spatial relations. After an introduction of the basic ideas and notions of topology, a novel theory of topological spatial relations between sets is developed in which the relations are defined in terms of the intersections of the boundaries and interiors of two sets. By considering *empty* and *non-empty* as the values of the intersections, a total of sixteen topological spatial relations are described, each of which can be realized in $\mathbb{R}^2$. This set is reduced to nine relations if the sets are restricted to spatial regions, a fairly broad class of subsets of a connected topological space having application to GIS. It is shown that these relations correspond to some of the standard set-theoretic and topological spatial relations between sets such as equality, disjointness, and containment in the interior.

1. Introduction

The present investigations have been motivated by the practical need for a formal understanding of spatial relations within the realm of Geographic Information Systems (GISs). In order to display, process, or analyze spatial information, users select data from a GIS by asking queries. Almost any GIS query is based on spatial concepts. Many queries explicitly incorporate *spatial relations* to describe constraints about spatial objects to be analyzed

*This paper was published in the *International Journal for Geographical Information Systems*, 5(2):161–174, 1991.

or displayed. For example, a GIS user may ask the following query to get information about potential risks of toxic waste dumps on school children in a specific area:

> Retrieve all toxic waste dumps which are within 10 miles of an elementary school and located in Penobscot County and its adjacent counties.

The number of elementary schools known to the information system are restricted by using the formulation of constraints. Of particular interest are the spatial constraints expressed by *spatial relations* such as *in*, *adjacent*, and *within 10 miles*.

The lack of a comprehensive theory of spatial relations has been a major impediment to any GIS implementation. The problem is not only one of selecting the appropriate terminology for these spatial relations, but rather one of determining their semantics. The development of a theory of spatial relations is expected to provide answers to the following questions (Abler, 1987):

- What are the fundamental geometric properties of geographic objects to describe their relations?
- How can these relations be formally defined in terms of fundamental geometric properties?
- What is a minimal set of spatial relations?

Besides the purely mathematical aspects, cognitive, linguistic, and psychological considerations (Talmy, 1983; Herskovits, 1986) must also be included if a theory about spatial relations, applicable to real-world problems, is to be developed (NCGIA, 1989). Within the scope of this paper we will focus only on the formal, mathematical concepts which have been partially provided from point-set topology.

The application of such a theory of spatial relations exceeds the domain of GISs. Any branch of science and engineering that deals with spatial data will benefit from a formal understanding of spatial relations. In particular, its contribution to a spatial logic and spatial reasoning will also be helpful in areas such as surveying engineering, CAD/CAM, robotics, and VLSI design.

The variety of spatial relations can be grouped into three different categories:

- topological relations which are invariant under topological transformations of the reference objects (Egenhofer, 1989; Egenhofer and Herring, 1990);
- metric relations in terms of distances and directions (Peuquet and Ci-Xiang, 1987); and
- relations concerning the partial and total order of spatial objects (Kainz, 1990) as described by prepositions such as *in front of*, *behind*, *above*, and *below* (Freeman, 1975; Chang *et al.*, 1989; Hernández, 1991).

Within the scope of this paper, we are only interested in the topological spatial relations.

In the past, formalisms for relations have been limited to simple data types in a one-dimensional space such as integers, reals, or their combinations, e.g., as intervals (Allen, 1983). Spatial data, such as geographic objects or CAD/CAM models, extend in higher dimensions. It has been assumed that a set of primitive relations in such a space is richer, but so far no attempt has been made to systematically explore this assumption.

The goal of this paper is twofold: First, to show that the description of topological spatial relations in terms of topologically invariant properties of point-sets is rather simple. As a consequence, the topological spatial relation between two point-sets may be determined with little computational effort. Second, we want to show that there exists a framework within which any topological spatial relation falls. This does not state that the set of relations determined by this formalism is complete—i.e., humans may distinguish additional relations—but that the formalism provides a complete coverage, i.e., any such additional relation will be only a specialization of one of the relations described.

As the underlying data model, we selected subsets of a topological space. The point-set approach is the most general model for the representation of topological spatial regions. Other approaches to the definition of topological spatial relations using different models, such as intervals (Pullar and Egenhofer, 1988) or simplicial complexes (Egenhofer, 1989), are generalized by our point-set approach.

This paper is organized as follows: the next section reviews previous approaches to defining topological spatial relations. Section 3. summarizes the relevant concepts of point-set topology and introduces the notions used in the remainder of the paper. Section 4. introduces the definition of topological spatial relations and shows their realization in $\mathbb{R}^2$. Section 5. investigates the existence of the relations between two spatial regions, subsets of a topological space with particular application to geographic data handling. In Section 6., the relations within $\mathbb{R}^n (n \geq 2)$ and $\mathbb{R}^1$ are compared.

2. Previous Work

Various collections of terms for spatial relations can be found in the computer science and geography literature (Freeman, 1975; Claire and Guptill, 1982; Chang *et al.*, 1989; Molenaar, 1989). In particular, designs of spatial query languages (Frank, 1982; Ingram and Phillips, 1987; Smith *et al.*, 1987; Herring *et al.*, 1988; Roussopoulos *et al.*, 1988) are a reservoir for informal notations of spatial relations with verbal explanations in natural language. A major drawback of these terms is the lack of a formal underpinning, because their definitions are frequently based on other expressions which are not exactly defined, but are assumed to be generally understood.

Most formal definitions of spatial relations describe them as the results of binary point-set operations. The subsequent review of these approaches will show their advantages and deficiencies. It will be obvious that none of the previous studies has been performed systematically enough to be used as a means to prove that the relations defined provide a complete coverage for the topological spatial relations between two spatial objects. Some definitions consider only a limited subset of representations of "spatial objects," while others apply insufficient concepts to define the whole range of topological spatial relations.

A formalism using the primitives *distance* and *direction* in combination with the logical connectors *AND*, *OR*, and *NOT* (Peuquet, 1986) will not be considered here. The assumption that every space has a metric is obviously too restrictive so that this formalism cannot be applied in a purely topological setting.

The definitions of relations in terms of set operations use pure set theory to describe topological relations. For example, the following definitions based on point-sets have been given for *equal*, *not equal*, *inside*, *outside*, and *intersects* in terms of the set operations $=$, $\neq$, $\subseteq$, and $\cap$ (Güting, 1988):

$$\begin{aligned}
x = y \quad &:= \quad \text{points } (x) = \text{ points } (y) \\
x \neq y \quad &:= \quad \text{points } (x) \neq \text{ points } (y) \\
x \text{ inside } y \quad &:= \quad \text{points } (x) \subseteq \text{ points } (y) \\
x \text{ outside } y \quad &:= \quad \text{points } (x) \cap \text{ points } (y) = \emptyset \\
x \text{ intersects } y \quad &:= \quad \text{points } (x) \cap \text{ points } (y) \neq \emptyset
\end{aligned}$$

The drawback of these definitions is that this set of relations is neither orthogonal nor complete. For instance, *equal* and *inside* are both covered by the definition of *intersects*. On the other hand, the model of point-sets *per se* does not allow for the definition of those relations that are based on the distinction of particular parts of the point-sets such as the boundary and the interior. For example, the relation *intersects* is topologically different from the one where common boundary points exist, but no common interior points are encountered.

The point-set approach has been augmented with the consideration of *boundary* and *interior* so that *overlap* and *neighbor* can be distinguished (Pullar, 1988):

$$\begin{aligned}
x \text{ overlaps } y \quad &:= \quad \text{boundary } (x) \cap \text{ boundary } (y) \neq \emptyset \text{ and} \\
& \text{interior } (x) \cap \text{ interior } (y) \neq \emptyset \\
x \text{ neighbor } y \quad &:= \quad \text{boundary } (x) \cap \text{ boundary } (y) \neq \emptyset \text{ and} \\
& \text{interior } (x) \cap \text{ interior } (y) = \emptyset
\end{aligned}$$

In a more systematic approach, boundaries and interiors have been identified as the crucial descriptions of polygonal intersections (Wagner, 1988). By comparing whether or not boundaries and interiors intersect, four relations have been identified:

- *neighborhood* where boundaries intersect, but interiors do not;

- *separation* where neither boundaries nor interiors intersect;
- *strict inclusion* where the boundaries do not intersect, but the interiors do; and
- *intersection* with both boundaries and interiors intersecting.

This approach uses a single, coherent method for the description of topological spatial relations, but it is not carried out in all its consequences. For example, no distinction can be made between *intersection* and *equality*, because for both relations boundaries and interiors intersect.

3. Point-Set Topology

Our model of topological spatial relations is based on the point-set topological notions of *interior* and *boundary*. In this section we will present the appropriate definitions and results from point-set topology. Some of the results are stated without proofs. Those proofs are all straightforward consequences of the definitions and can be found in most basic topology text books, e.g., by Munkres (1966) and Spanier (1966).
Let X be a set. A *topology* on X is a collection $\mathcal{A}$ of subsets of X that satisfies the three conditions:

- the empty set and X are in $\mathcal{A}$,
- $\mathcal{A}$ is closed under arbitrary unions, and
- $\mathcal{A}$ is closed under finite intersections.

A *topological space* is a set X with a topology $\mathcal{A}$ on X. The sets in a topology on X are called *open sets*, and their complements in X are called *closed sets*. The collection of closed sets

- contains the empty set and X,
- is closed under arbitrary intersections, and
- is closed under finite unions.

Via the open sets in a topology on a set X, a set-theoretic notion of closeness is established. If U is an open set and $x \in U$, then U is said to be a *neighborhood* of x. This set-theoretic notion of closeness generalizes the metric notion of closeness. A metric d on a set X induces a topology on X, called the *metric topology defined by* d. This topology is such that $U \subset X$ is an open set if for each $x \in U$, there is an $\varepsilon > 0$ such that the d-ball† of radius ε around x is contained in U.
For the remainder of this paper let X be a set with a topology $\mathcal{A}$. If S is a subset of X then S inherits a topology from $\mathcal{A}$. This topology is called the *subspace topology* and is defined such that $U \subset S$ is open in the subspace topology if and only if $U = S \cap V$ for some set $V \in \mathcal{A}$. Under such circumstances, S is called a *subspace* of X.

†A d-ball is the set of points whose distance from x in the metric d is less than ε, i.e., $\{y \in X \,|\, d(x, y) < \varepsilon\}$.

3.1 Interior

Given $Y \subset X$, the *interior* of Y, denoted by Y°, is defined to be the union of all open sets that are contained in Y, i.e., the interior of Y is the largest open set contained in Y. y is in the interior of Y if and only if there is a neighborhood of y contained in Y, i.e., $y \in Y^\circ$ if and only if there is an open set U such that $y \in U \subset Y$. The interior of a set could be empty, e.g., the interior of the empty set is empty. The interior of X is X itself. If U is open then $U^\circ = U$. If $Z \subset Y$ then $Z^\circ \subset Y^\circ$.

3.2 Closure

The *closure* of Y, denoted by $\overline{Y}$, is defined to be the intersection of all closed sets that contain Y, i.e., the closure of Y is the smallest closed set containing Y. It follows that y is in the closure of Y if and only if every neighborhood of y intersects Y, i.e., $y \in \overline{Y}$ if and only if $U \cap Y \neq \emptyset$ for every open set U containing y. The empty set is the only set with empty closure. The closure of X is X itself. If C is closed then $\overline{C} = C$. If $Z \subset Y$ then $\overline{Z} \subset \overline{Y}$.

3.3 Boundary

The *boundary* of Y, denoted by ∂Y, is the intersection of the closure of Y and the closure of the complement of Y, i.e., $\partial Y = \overline{Y} \cap \overline{X - Y}$. The boundary is a closed set. It follows that y is in the boundary of Y if and only if every neighborhood of y intersects both Y and its complement, i.e., $y \in \partial Y$ if and only if $U \cap Y \neq \emptyset$ and $U \cap (X - Y) \neq \emptyset$ for every open set U containing y. The boundary can be empty, e.g., the boundaries of both X and the empty set are empty.

3.4 Relationship between Interior, Closure, and Boundary

The concepts of interior, closure, and boundary are fundamental to the forthcoming discussions of topological spatial relations between sets. The relationships between interior, closure, and boundary are described by the following propositions:

Proposition 3.1 $Y^\circ \cap \partial Y = \emptyset$.

Proof: If $x \in \partial Y$, then every neighborhood U of x intersects $X - Y$ so that U cannot be contained in Y. Since no neighborhood U of x is contained in Y it follows that $x \notin Y^\circ$ and, therefore, $\partial Y \cap Y^\circ = \emptyset$. □

Proposition 3.2 $Y^\circ \cup \partial Y = \overline{Y}$.

Proof: $Y^\circ \subset Y \subset \overline{Y}$ and, by definition, $\partial Y \subset \overline{Y}$. Since Y° and ∂Y are both subsets of $\overline{Y}$ it follows that $(Y^\circ \cup \partial Y) \subset \overline{Y}$. To show that $\overline{Y} \subset (Y^\circ \cup \partial Y)$, let $x \in \overline{Y}$ and assume that $x \notin Y^\circ$. We show that $x \in \partial Y$ which, since $x \in \overline{Y}$, only requires showing that $x \in \overline{X - Y}$. $x \notin Y^\circ$ implies that every neighborhood of x is not contained in Y; therefore, every neighborhood of x intersects $X - Y$, implying that $x \in \overline{X - Y}$. So $x \in \partial Y$. Thus if $x \in \overline{Y}$ and $x \notin Y^\circ$ then $x \in \partial Y$ and it follows that $\overline{Y} \subset (Y^\circ \cup \partial Y)$. Thus $\overline{Y} = (Y^\circ \cup \partial Y)$. □

3.5 Separation

The concepts of separation and connectedness are crucial for establishing the forthcoming topological spatial relations between sets. Let $Y \subset X$. A *separation* of Y is a pair A, B of subsets of X satisfying the following three conditions:

- $A \neq \emptyset$ and $B \neq \emptyset$;
- $A \cup B = Y$; and
- $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$.

If there exists a separation of Y then Y is said to be *disconnected*, otherwise Y is said to be *connected*. If Y is the union of two non-empty disjoint open subsets of X then it follows that Y is disconnected. If C is connected and $C \subset D \subset \overline{C}$ then D is connected. In particular, if C is connected then $\overline{C}$ is connected; however, ∂C and C° need not be connected.

Proposition 3.3 *If A, B form a separation of Y and if Z is a connected subset of Y then either $Z \subset A$ or $Z \subset B$.*

Proof: By assumption, Z is a subset of the union of A and B, i.e., $Z \subset A \cup B$. We will show that the intersection between Z and one of A or B is empty, i.e., either $Z \cap B = \emptyset$ or $Z \cap A = \emptyset$. Suppose not, i.e., assume that both intersections are non-empty. Let $C = Z \cap A$ and $D = Z \cap B$. Then C and D are both non-empty and $C \cup D = Z$. Since $\overline{C} \subset \overline{A}$, $D \subset B$, and $\overline{A} \cap B = \emptyset$ (because A, B is a separation of Y), it follows that $\overline{C} \cap D = \emptyset$. Similarly, $C \cap \overline{D} = \emptyset$; therefore, C and D form a separation of Z, contradicting the assumption that Z is connected. So either $Z \cap B = \emptyset$ or $Z \cap A = \emptyset$, implying that either $Z \subset A$ or $Z \subset B$. □

A subset Z of X is said to *separate* X if $X - Z$ is disconnected. The following separation result gives simple conditions under which the boundary of a subset of X separates X.

Proposition 3.4 *Assume $Y \subset X$. If $Y^\circ \neq \emptyset$ and $\overline{Y} \neq X$, then Y° and $X - \overline{Y}$ form a separation of $X - \partial Y$, and thus ∂Y separates X.*

Proof: By assumption, Y° and $X - \overline{Y}$ are non-empty. Clearly, they are disjoint open sets. Proposition 3.2 implies that $X - \partial Y = Y^\circ \cup (X - \overline{Y})$. It follows that Y° and $X - \overline{Y}$ form a separation of $X - \partial Y$. □

3.6 Topological Equivalence

The study of topological equivalence is central to the theory of topology. Two topological spaces are *topologically equivalent*[‡] if there is a bijective function between them that yields a bijective correspondence between the open sets in the respective topologies. Such a function, which is continuous with a continuous inverse, is called a *homeomorphism*. Examples of

[‡] also: are *homeomorphic* or *of the same topological type*.

homeomorphisms are the Euclidean notions of translation, rotation, scale, and skew. Properties of topological spaces that are preserved under homeomorphism are called *topological invariants* of the spaces. For example, the property of connectedness is a topological invariant.

4. A Framework for the Description of Topological Spatial Relations

Our model describing the topological spatial relations between two subsets, A and B, of a topological space X is based on a consideration of the four intersections of the boundaries and interiors of the two sets A and B, i.e., $\partial A \cap \partial B$, $A^\circ \cap B^\circ$, $\partial A \cap B^\circ$, and $A^\circ \cap \partial B$.

Definition 4.1 *Let A, B be a pair of subsets of a topological space X. A topological spatial relation between A and B is described by a four-tuple of values of topological invariants associated respectively to each of the four sets $\partial A \cap \partial B$, $A^\circ \cap B^\circ$, $\partial A \cap B^\circ$, and $A^\circ \cap \partial B$.*

A topological spatial relation between two sets is preserved under homeomorphism of the underlying space X. Specifically, if $f : X \to Y$ is a homeomorphism and $A, B \subset X$, then $\partial A \cap \partial B$, $A^\circ \cap B^\circ$, $\partial A \cap B^\circ$, and $A^\circ \cap \partial B$ are mapped homeomorphically onto $\partial f(A) \cap \partial f(B)$, $f(A)^\circ \cap f(B)^\circ$, $\partial f(A) \cap f(B)^\circ$, and $f(A)^\circ \cap \partial f(B)$, respectively. Since the topological spatial relation is defined in terms of topological invariants of these intersections, it follows that the topological spatial relation between A and B in X is identical to the topological spatial relation between $f(A)$ and $f(B)$ in Y.

We denote a topological spatial relation by a four-tuple $(\_, \_, \_, \_)$. The entries correspond in order to the values of topological invariants associated to the four set-intersections. We will call the first intersection the *boundary-boundary* intersection, the second intersection the *interior-interior* intersection, the third intersection the *boundary-interior* intersection, and the fourth intersection the *interior-boundary* intersection.

4.1 Topological Spatial Relations from Empty/Non-Empty Set-Intersections

As the entries in the four-tuple, we consider properties of sets that are invariant under homeomorphisms. For example, the properties *empty* and *non-empty* are set-theoretic, and therefore topologically, invariant. Other invariants, not considered in this paper, are the dimension of a set and the number of connected components (Munkres, 1966). Empty/non-empty is the simplest and most general invariant so that any other invariant may be considered a more restrictive classifier.

For the remainder of this paper, we restrict our attention to the binary topological spatial relations defined by assigning the appropriate value of *empty* ($\emptyset$) and *non-empty* ($\neg\emptyset$) to the entries in the four-tuple. The sixteen possibilities from these combinations are summarized in table 1.

	$\partial \cap \partial$	$^\circ \cap ^\circ$	$\partial \cap ^\circ$	$^\circ \cap \partial$
r_0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
r_1	$\neg\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
r_2	$\emptyset$	$\neg\emptyset$	$\emptyset$	$\emptyset$
r_3	$\neg\emptyset$	$\neg\emptyset$	$\emptyset$	$\emptyset$
r_4	$\emptyset$	$\emptyset$	$\neg\emptyset$	$\emptyset$
r_5	$\neg\emptyset$	$\emptyset$	$\neg\emptyset$	$\emptyset$
r_6	$\emptyset$	$\neg\emptyset$	$\neg\emptyset$	$\emptyset$
r_7	$\neg\emptyset$	$\neg\emptyset$	$\neg\emptyset$	$\emptyset$
r_8	$\emptyset$	$\emptyset$	$\emptyset$	$\neg\emptyset$
r_9	$\neg\emptyset$	$\emptyset$	$\emptyset$	$\neg\emptyset$
r_{10}	$\emptyset$	$\neg\emptyset$	$\emptyset$	$\neg\emptyset$
r_{11}	$\neg\emptyset$	$\neg\emptyset$	$\emptyset$	$\neg\emptyset$
r_{12}	$\emptyset$	$\emptyset$	$\neg\emptyset$	$\neg\emptyset$
r_{13}	$\neg\emptyset$	$\emptyset$	$\neg\emptyset$	$\neg\emptyset$
r_{14}	$\emptyset$	$\neg\emptyset$	$\neg\emptyset$	$\neg\emptyset$
r_{15}	$\neg\emptyset$	$\neg\emptyset$	$\neg\emptyset$	$\neg\emptyset$

Table 1: The sixteen specifications of binary topological relations based upon the criteria of empty and non-empty intersections of boundaries and interiors.

A set is either empty or non-empty; therefore, it is clear that these sixteen topological spatial relations provide complete coverage, that is, given any pair of sets A and B in X, there is always a topological spatial relation associated with A and B. Furthermore, a set cannot simultaneously be empty and non-empty, from which follows that the sixteen topological spatial relations are mutually exclusive, i.e., for any pair of sets A and B in X, exactly one of the sixteen topological spatial relations holds true.
In general, each of the sixteen spatial relations can occur between two sets. Depending on various restrictions on the sets and the underlying topological space, the actual set of existing topological spatial relations may be a subset of the sixteen in table 1. For general point-sets in the plane $\mathbb{R}^2$, all sixteen topological spatial relations can be realized (figure 1).

4.2 Influence of the Topological Space on the Relations

The setting, i.e., the topological space X in which A and B lie, plays an important role in the spatial relation between A and B. For example, in figure 2a the two sets A and B have the relation $(\emptyset, \neg\emptyset, \neg\emptyset, \neg\emptyset)$ as subsets of the line. The same configuration shows a different relation between the two sets when they are embedded in the plane (figure 2b). As subsets of the plane, the boundaries of A and B are equal to A and B, respectively, and the interiors are empty, i.e., $\partial A = A$, $A^\circ = \emptyset$, $\partial B = B$, and $B^\circ = \emptyset$. It follows that in the plane the spatial relation between the two sets A and B is $(\neg\emptyset, \emptyset, \emptyset, \emptyset)$.

5. Topological Relations between Spatial Regions

It is our aim to model topological spatial relations that occur between polygonal areas in the plane; therefore, we restrict the topological space X and the sets under consideration in X. Our restrictions are not too specific and the only assumption that we make about the topological space X is that it is connected. This guarantees that the boundary of each set of interest is not empty.

The sets of interest are the *spatial regions*, defined as follows:

Definition 5.1 *Let X be a connected topological space. A* spatial region *in X is a non-empty proper subset A of X satisfying (1) A° is connected and (2) $A = \overline{A^\circ}$.*

It follows from the definition that the interior of each spatial region is non-empty. Furthermore, a spatial region is closed and connected since it is the closure of a connected set.

The following proposition implies that the boundary of each spatial region is non-empty.

Proposition 5.2 *If A is a spatial region in X then $\partial A \neq \emptyset$.*

Proof: $A^\circ \neq \emptyset$. $A = \overline{A}$ since A is closed, and $A \neq X$ by definition of a spatial region. From proposition 3.4 it follows that A° and $X - A$ form a separation of $X - \partial A$. If $\partial A = \emptyset$ then the two sets would form a separation of X, which is impossible since X is connected; therefore, $\partial A \neq \emptyset$. □

5.1 Existence of Region Relations

The framework for the spatial relations between point-sets carries over to spatial regions, however, not all of the sixteen relations between arbitrary point-sets exist between two spatial regions. From the examples in figure 1 we conclude that at least the relations r_0, r_1, r_3, r_6, r_7, r_{10}, r_{11}, r_{14}, and r_{15} exist between two spatial regions. The following proposition shows that these nine topological spatial relations are the only ones that can occur between spatial regions.

Proposition 5.3 *For two spatial regions the spatial relations r_2, r_4, r_5, r_8, r_9, r_{12}, and r_{13} cannot occur.*

Proof: We begin by proving that if the boundary-interior or interior-boundary intersection is non-empty then the interior-interior intersection between the same two regions is non-empty as well. This implies that the six topological spatial relations r_4, r_5, r_8, r_9, r_{12}, and r_{13}, all with empty interior-interior and non-empty boundary-interior or interior-boundary intersections, cannot occur.

Let A and B be spatial regions for which $\partial A \cap B^\circ \neq \emptyset$. We show that $A^\circ \cap B^\circ \neq \emptyset$. Using proposition 3.2, we have $A^\circ \cup \partial A = \overline{A}$ and $A^\circ \cup \partial(A^\circ) = \overline{A^\circ}$. $\overline{A} = A = \overline{A^\circ}$, so $A^\circ \cup \partial A = A^\circ \cup \partial(A^\circ)$. Furthermore, by proposition 3.1, $A^\circ \cap \partial(A^\circ) = \emptyset$ and $A^\circ \cap \partial A = \emptyset$. It follows that

$\partial(A^\circ) = \partial A$. Now let $x \in \partial A \cap B^\circ$, then $x \in \partial(A^\circ)$, and since B° is open and contains x, it follows that $A^\circ \cap B^\circ \neq \emptyset$. Thus if the boundary-interior intersection is non-empty then the interior-interior intersection is non-empty as well. It also follows that if the interior-boundary intersection is non-empty then the interior-interior intersection is also non-empty.
Next we prove that if the boundary-boundary intersection is empty and the interior-interior intersection is not empty then either the boundary-interior or the interior-boundary intersection is not empty. This implies that the spatial relation r_2, with a non-empty interior-interior intersection and empty intersections for boundary-boundary, boundary-interior, and interior-boundary, cannot occur. This will complete the proof of the proposition.
Let A and B be spatial regions such that $\partial A \cap \partial B = \emptyset$ and $A^\circ \cap B^\circ \neq \emptyset$. We will show that if $\partial A \cap B^\circ = \emptyset$ then $A^\circ \cap \partial B \neq \emptyset$. Assume that $\partial A \cap B^\circ = \emptyset$. Since $B = B^\circ \cup \partial B$ it follows that $\partial A \cap B = \emptyset$ and, therefore, $B \subset X - \partial A$. Proposition 3.4 implies that A° and $X - A$ form a separation of $X - \partial A$, and since B is connected, proposition 3.3 implies that either $B \subset A^\circ$ or $B \subset X - A$. Since, by assumption, $A^\circ \cap B^\circ \neq \emptyset$ it follows that $B \subset A^\circ$ and, therefore, $\partial B \subset A^\circ$. Clearly, $\partial B \cap A^\circ \neq \emptyset$ and the result follows. □

5.2 Semantics of Region Relations

In figure 1, examples were depicted for the topological spatial relations r_0, r_1, r_3, r_6, r_7, r_{10}, r_{11}, r_{14}, and r_{15} between spatial regions. We consider each of these nine relations in the definitions below and will investigate their semantics using the same notation as in (Egenhofer, 1989) and (Egenhofer and Herring, 1990).

	$\partial \cap \partial$	$^\circ \cap ^\circ$	$\partial \cap ^\circ$	$^\circ \cap \partial$	
r_0	$(\emptyset,$	$\emptyset,$	$\emptyset,$	$\emptyset)$	A and B are disjoint
r_1	$(\neg\emptyset,$	$\emptyset,$	$\emptyset,$	$\emptyset)$	A and B touch
r_3	$(\neg\emptyset,$	$\neg\emptyset,$	$\emptyset,$	$\emptyset)$	A equals B
r_6	$(\emptyset,$	$\neg\emptyset,$	$\neg\emptyset,$	$\emptyset)$	A is inside of B or B contains A
r_7	$(\neg\emptyset,$	$\neg\emptyset,$	$\neg\emptyset,$	$\emptyset)$	A is covered by B or B covers A
r_{10}	$(\emptyset,$	$\neg\emptyset,$	$\emptyset,$	$\neg\emptyset)$	A contains B or B is inside of A
r_{11}	$(\neg\emptyset,$	$\neg\emptyset,$	$\emptyset,$	$\neg\emptyset)$	A covers B or B is covered by A
r_{14}	$(\emptyset,$	$\neg\emptyset,$	$\neg\emptyset,$	$\neg\emptyset)$	A and B overlap with disjoint boundaries
r_{15}	$(\neg\emptyset,$	$\neg\emptyset,$	$\neg\emptyset,$	$\neg\emptyset)$	A and B overlap with intersecting boundaries

Table 2: The terminology used for the nine relations between two spatial regions.

Definition 5.4 *The descriptive terms for the nine topological spatial relations between two regions are given in table 2.*

If the topological spatial relation between A and B is r_0 then, in the set-theoretic sense, A and B are disjoint and, therefore, the topological spatial relation *disjoint* coincides with the set-theoretic notion of disjoint. The following proposition and corollaries justify the other descriptive terms for the topological spatial relations defined in table 2.

Proposition 5.5 *Let A and B be spatial regions in X. If $A^\circ \cap B^\circ \neq \emptyset$ and $A^\circ \cap \partial B = \emptyset$ then $A^\circ \subset B^\circ$ and $A \subset B$.*

Proof: A° is connected. Proposition 3.4 implies that B° and $X - B$ form a separation of X. Since $A^\circ \cap \partial B = \emptyset$ it follows by proposition 3.1 that $A^\circ \subset B^\circ \cup (X - B)$. Proposition 3.3 implies that either $A^\circ \subset B^\circ$ or $A^\circ \subset (X - B)$. But $A^\circ \cap B^\circ \neq \emptyset$; therefore, $A^\circ \subset B^\circ$. Since $A^\circ \subset B^\circ$ it follows that $\overline{A^\circ} \subset \overline{B^\circ}$ which, by definition 5.1, implies that $A \subset B$. □

From proposition 5.5 follows that if A is covered by B then $A \subset B$; therefore, the spatial relation *is covered by* coincides with the set-theoretic notion of being a subset of.

The following corollary to proposition 5.5 shows that the spatial relation *equal* corresponds to the set-theoretic notion of equality.

Corollary 5.6 *Let A and B be spatial regions. If the spatial relation between A and B is r_3 then $A = B$.*

Proof: $A^\circ \cap B^\circ \neq \emptyset$ and $A^\circ \cap \partial B = \emptyset$; therefore, proposition 5.5 implies that $A \subset B$. Furthermore, $\partial A \cap B^\circ = \emptyset$. Again by proposition 5.5, $B \subset A$. Thus $A = B$. □

The following corollary to proposition 5.5 shows that if A is inside of B then $A \subset B^\circ$; therefore, the spatial relation *inside* coincides with the topological notion of being contained in the interior. Conversely, *contains* corresponds to contains in the interior.

Corollary 5.7 *Let A and B be spatial regions. If the spatial relation between A and B is r_6 then $A \subset B^\circ$.*

Proof: Proposition 5.5 implies that $A^\circ \subset B^\circ$ and $A \subset B$. By proposition 3.2, $A = A^\circ \cup \partial A$ and $B = B^\circ \cup \partial B$. So $\partial A \subset B$. Since $\partial A \cap \partial B = \emptyset$ it follows that $\partial A \subset B^\circ$. Together with $A^\circ \subset B^\circ$ this implies that $A \subset B^\circ$. □

6. Relations in n-Dimensional Spaces

It is natural to ask "What further restrictions on the topological space X and the sets under consideration in X further reduces the topological spatial relations that can occur?" This section will explore this question by considering the case where X is a Euclidean space.

$\mathbb{R}^n$ denotes n-dimensional Euclidean space with the usual Euclidean metric. A subset of $\mathbb{R}^n$ is *bounded* if there is an upper bound to the distances between pairs of points in the set; otherwise, it is said to be *unbounded*.

The *unit disk* in $\mathbb{R}^n$ is the set of points in $\mathbb{R}^n$ whose distance from the origin is less than or equal to 1.
The *unit sphere* in $\mathbb{R}^n$ is the set of points in $\mathbb{R}^n$ whose distance from the origin is equal to 1. For $n \geq 1$ the unit disk in $\mathbb{R}^n$ is connected. For $n \geq 2$ the unit sphere in $\mathbb{R}^n$ is connected. Let X be a topological space. An ***n*-disk** in X is a subspace of X that is homeomorphic to the unit disk in $\mathbb{R}^n$. An *n-sphere* in X is a subspace of X that is homeomorphic to the unit sphere in $\mathbb{R}^{n+1}$. n-disks in $\mathbb{R}^n$ are bounded and are spatial regions; the latter is a relatively straightforward consequence of the Brower theorem on the invariance of domain (Spanier, 1966). Since n-disks in $\mathbb{R}^n$ are spatial regions, proposition 5.3 restricts the number of spatial relations that can occur between them.
In proposition 6.1 we show that if A and B are n-disks in $\mathbb{R}^n$ with $n \geq 2$ then the spatial relation *overlap with disjoint boundary* cannot occur. The proof of this proposition is based on the following two facts:

Fact 1 *Let A be an n-disk in $\mathbb{R}^n$ with $n \geq 2$. Then ∂A is an $(n-1)$-sphere in $\mathbb{R}^n$ and, therefore, connected.*

This fact, also, is a consequence of the Brower theorem on the invariance of domain (Spanier, 1966).

Fact 2 *Let A be an n-disk in $\mathbb{R}^n$ with $n \geq 2$. Then $\mathbb{R}^n - A^\circ$ is connected and unbounded.*

This second fact is a (non-)separation theorem related to the Jordan-Brower separation theorem (Spanier, 1966).

Proposition 6.1 *The topological spatial relation* r$_{14}$, overlap with disjoint boundaries, *does not occur between n-disks in $\mathbb{R}^n$ with $n \geq 2$.*

Proof: Let A and B be n-disks in $\mathbb{R}^n$ with $n \geq 2$. We show that if $\partial A \cap \partial B = \emptyset$ then A and B do not overlap and, therefore, the spatial relation *overlap with disjoint boundary* cannot occur.
Assume $\partial A \cap \partial B = \emptyset$ and A and B overlap. We will derive a contradiction. B is a spatial region; therefore, proposition 3.4 implies that B° and $\mathbb{R}^n - B$ form a separation of $\mathbb{R}^n - \partial B$. Since $\partial A \cap \partial B = \emptyset$ it follows that $\partial A \subset \mathbb{R}^n - \partial B$. By fact 1, ∂A is connected, therefore, proposition 3.3 implies that either $\partial A \subset B^\circ$ or $\partial A \subset (\mathbb{R}^n - B)$. Since A and B overlap, it follows that $\partial A \cap B^\circ \neq \emptyset$ and, therefore, $\partial A \subset B^\circ$.
$\partial A \subset B^\circ$ implies that $\partial A \cap (\mathbb{R}^n - B^\circ) = \emptyset$. By fact 2, $\mathbb{R}^n - B^\circ$ is connected. Using propositions 3.3 and 3.4 and arguing as above, it follows that either $(\mathbb{R}^n - B^\circ) \subset A^\circ$ or $(\mathbb{R}^n - B^\circ) \subset (\mathbb{R}^n - A)$. The first case yields a contradiction, because, by fact 2, $\mathbb{R}^n - B^\circ$ is unbounded, but A° is not. The second case implies that $A \subset B^\circ$ and, therefore, $A^\circ \cap \partial B = \emptyset$ which contradicts the assumption that A and B overlap. So, in either case we get a contradiction and it follows that the spatial relation r$_{14}$ cannot occur between n-disks in $\mathbb{R}^n$ with $n \geq 2$. □

Note that for $n \geq 2$ the topological spatial relation r_{15}, *overlap with intersecting boundaries*, does occur between two n-disks (figure 1).
The opposite situation occurs in $\mathbb{R}^1$ where r_{14} can occur between 1-disks, while r_{15}, *overlap with intersecting boundaries*, cannot. It is clear that r_{14} can occur between two 1-disks in $\mathbb{R}^1$ (figure 2). Proposition 6.2 shows that r_{15} cannot occur. Its proof requires the easily-derived fact that a spatial region in $\mathbb{R}^1$ is either a closed interval $[a, b]$ for some $a, b \in R^1$, or a closed ray $[a, \infty)$ or $(-\infty, a]$ for some $a \in \mathbb{R}^1$.

Proposition 6.2 *The topological spatial relation* r_{15} *does not occur between spatial regions in* $\mathbb{R}^1$.

Proof: Let A and B be spatial regions in $\mathbb{R}^1$ and assume that A and B overlap. We show that $\partial A \cap \partial B = \emptyset$. Each of A and B is a closed interval or a closed ray; therefore, we have nine different cases to examine. We select one, the others can be proven accordingly.
Assume $A = [a, \infty)$ and $B = (-\infty, b]$. Then $\partial A = \{a\}$ and $\partial B = \{b\}$. Since A and B overlap, it follows that $a < b$ which implies that $\partial A \cap \partial B = \emptyset$. $\square$

7. Conclusions

A framework for the definition of topological spatial relations has been presented. It is based upon purely topological properties and thus independent of the existence of a distance function. The topological relations are described by the four intersections of the boundaries and interiors of two point-sets. Considering the binary values empty and non-empty for these intersections a set of sixteen mutually exclusive specifications has been identified. Fewer relations exist if particular restrictions on the point-sets and the topological space are made. We proved that there are only nine topological spatial relations between point-sets which are homeomorphic to polygonal areas in the plane.
Though the nature of the present work is rather theoretical, the framework has immediate impact on the design and implementation of geographic information systems. Previously, for every topological spatial relation a separate procedure had to be programmed and no mechanism existed to assure completeness. Now, topological spatial relations can be derived from a single, consistent model and no programming for individual relations will be necessary. Prototype implementations of this framework have been designed and partially implemented (Egenhofer, 1989) and various extensions to the framework have been investigated to provide more details about topological spatial relations such as the consideration of the dimensions of the intersections and of the number of disconnected subsets in the intersections (Egenhofer and Herring, 1990). Ongoing investigations focus on the application of this framework for formal reasoning about combinations of topological spatial relationships.
The framework presented is considered a start and further investigations are necessary to verify its suitability. Here, only topological spatial relations

with codimension zero were considered, i.e, the difference between the dimension of the space and the dimension of the embedded spatial objects is zero, e.g., between regions in the plane and intervals on the one-dimensional line. Of interest for GIS applications are also the topological spatial relationships with codimension greater than zero, e.g., between two lines in the plane (Herring, 1991). Likewise, the applicability of this framework for topological spatial relations between objects of different dimensions, such as a region and a line, must be tested.

Acknowledgments

The motivation has been given by Bruce Palmer. During many discussions with John Herring, these concepts have been clarified. Andrew Frank and Renato Barrera made valuable comments on an earlier version of this paper.

References

Abler, R., 1987. The National Science Foundation National Center for Geographic Information and Analysis. *International Journal of Geographical Information Systems*, 1(4):303–326.

Allen, J., 1983. Maintaining Knowledge about Temporal Intervals. *Communications of the ACM*, 26(11):832–843, 1983.

Chang, S.K., Jungert, E., and Li, Y., 1989. The Design of Pictorial Databases Based Upon the Theory of Symbolic Projections. In: A. Buchmann, O. Günther, T. Smith, and Y. Wang, editors, *Symposium on the Design and Implementation of Large Spatial Databases, Lecture Notes in Computer Science, Vol. 409*, pages 303–323, Springer-Verlag, New York, NY.

Claire, R. and Guptill, S., 1982. Spatial Operators for Selected Data Structures. In: *Auto-Carto V*, pages 189–200, Crystal City, VA.

Egenhofer, M., 1989. A Formal Definition of Binary Topological Relationships. In: W. Litwin and H.-J. Schek, editors, *Third International Conference on Foundations of Data Organization and Algorithms (FODO), Paris, France, Lecture Notes in Computer Science, Vol. 367*, pages 457–472, Springer-Verlag, New York, NY.

Egenhofer, M. and Herring, J., 1990. A Mathematical Framework for the Definition of Topological Relationships. In: K. Brassel and H. Kishimoto, editors, *Fourth International Symposium on Spatial Data Handling*, pages 803–813, Zurich, Switzerland.

Frank, A., 1982. MAPQUERY—Database Query Language for Retrieval of Geometric Data and its Graphical Representation. *ACM Computer Graphics*, 16(3):199–207.

Freeman, J., 1975. The Modelling of Spatial Relations. *Computer Graphics and Image Processing*, 4:156–171.

Güting, R., 1988. Geo-Relational Algebra: A Model and Query Language for Geometric Database Systems. In: J. Schmidt, S. Ceri, and M. Missikoff, editors, *Advances in Database Technology—EDBT '88, International Conference on Extending Database Technology, Venice,* *Italy, Lecture Notes in Computer Science, Vol. 303*, pages 506–527, Springer-Verlag, New York, NY, 1988.

Hernández, D., 1991. Relative Representation of Spatial Knowledge: The 2-D Case. In: D. Mark and A. Frank, editors, *Cognitive and Linguistic Aspects of Geographic Space*, Kluwer Academic Publishers, Dordrecht (in press).

Herring, J., Larsen, R., and Shivakumar, J., 1988. Extensions to the SQL Language to Support Spatial Analysis in a Topological Data Base. In: *GIS/LIS '88*, pages 741–750, San Antonio, TX.

Herring, J., 1991. The Mathematical Modeling of Spatial and Non-Spatial Information in Geographic Information Systems. In: D. Mark and A. Frank, editors, *Cognitive and Linguistic Aspects of Geographic Space*, Kluwer Academic Publishers, Dordrecht (in press).

Herskovits, A., 1986. *Language and Spatial Cognition—An Interdisciplinary Study of the Prepositions in English*. Cambridge University Press, Cambridge.

Ingram, K. and Phillips, W., 1987. Geographic Information Processing Using a SQL-Based Query Language. In: N.R. Chrisman, editor, *AUTO-CARTO 8, Eighth International Symposium on Computer-Assisted Cartography*, pages 326–335, Baltimore, MD.

Kainz, W., 1990. Spatial Relationships—Topology versus Order. In: K. Brassel and H. Kishimoto, editors, *Fourth International Symposium on Spatial Data Handling*, pages 814–819, Zurich, Switzerland.

Molenaar, M., 1989. Single Valued Vector Maps—A Concept in Geographic Information Systems. *Geo-Information-Systems*, 2(1):18–26.

Munkres, J., 1966. *Elementary Differential Topology*. Princeton University Press, Princeton, NJ.

NCGIA, 1989. The Research Plan of the National Center for Geographic Information and Analysis. *International Journal of Geographical Information Systems*, 3(2):117–136.

Peuquet, D., 1986. The Use of Spatial Relationships to Aid Spatial Database Retrieval. In: D. Marble, editor, *Second International Symposium on Spatial Data Handling*, pages 459–471, Seattle, WA.

Peuquet, D. and Ci-Xiang, Z., 1987. An Algorithm to Determine the Directional Relationship Between Arbitrarily-Shaped Polygons in the Plane. *Pattern Recognition*, 20(1):65–74.

Pullar, D., 1988. Data Definition and Operators on a Spatial Data Model. In: *ACSM-ASPRS Annual Convention*, pages 197–202, St. Louis, MO.

Pullar, D. and Egenhofer, M., 1988. Towards Formal Definitions of Topological Relations Among Spatial Objects. In: D. Marble, editor, *Third International Symposium on Spatial Data Handling*, pages 225–242, Sydney, Australia.

Roussopoulos, N., Faloutsos, C., and Sellis, T., 1988. An Efficient Pictorial Database System for PSQL. *IEEE Transactions on Software Engineering*, 14(5):630–638.

Smith, T., Peuquet, D., Menon, S., and Agrawal, P., 1987. KBGIS-II: A Knowledge-Based Geographical Information System. *International Journal of Geographical Information Systems*, 1(2):149–172.

Spanier, E., 1966. *Algebraic Topology*. McGraw-Hill Book Company, New York, NY.

Talmy, L., 1983. How Language Structures Space. In: H. Pick and L. Acredolo, editors, *Spatial Orientation: Theory, Research, and Application*, pages 225–282, Plenum Press, New York, NY.

Wagner, D., 1988. A Method of Evaluating Polygon Overlay Algorithms. In: *ACSM-ASPRS Annual Convention*, pages 173–183, St. Louis, MO.

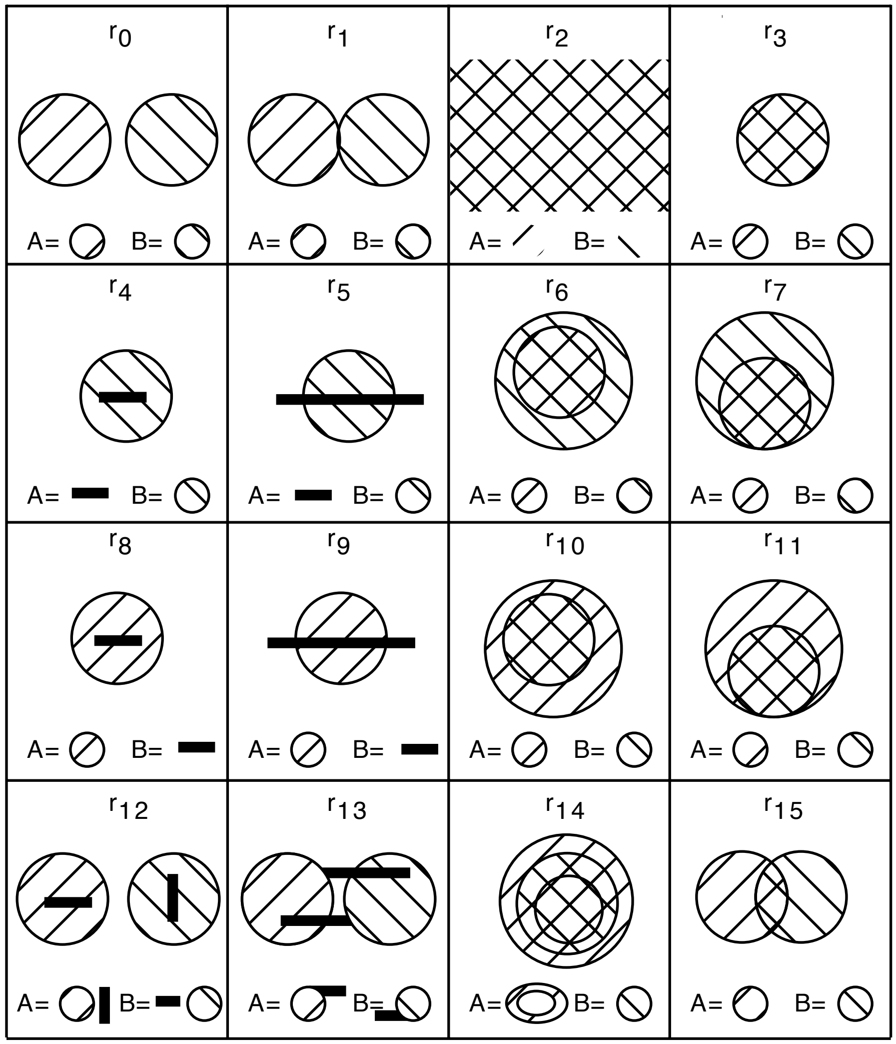

Figure 1. Examples of the 16 binary topological spatial relations based on the comparion of empty and non-empty set-intersections between boundaries and interiors.

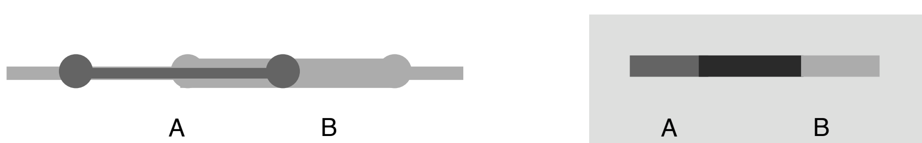

Figure 2. The same configuration of the two sets *A* and *B* with (a) the topological spatial relation (Ø, ¬Ø, ¬Ø, ¬Ø) when embedded in a line and (b) (¬Ø, Ø, Ø, Ø) in a plane.

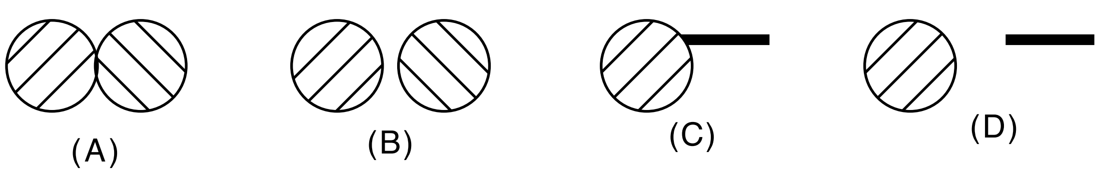

Figure 3. Sets in teh plane that are not spatial regions.