MINISTÉRIO DA CIÊNCIA E TECNOLOGIA
INSTITUTO NACIONAL DE PESQUISAS ESPACIAIS

AN ALGEBRA FOR OPENGIS® COVERAGES BASED ON TOPOLOGICAL PREDICATES

Danilo Mori Palomo

Master Thesis in Applied Computing Science, advised by
Dr. Gilberto Câmara

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ABSTRACT

Map algebra is a collection of functions for handling spatial datasets where each data contains a set of geometries of the same type which bound to a geographical reference. The current theory for Map Algebra uses *ad hoc* operators proposed by Dana Tomlin. His proposal has had great practical success, and most GIS implementations provide its operations. However, there is a lack of theoretical foundations for the operations proposed in Tomlin's map algebra. This is a limitation for the proposal of international standards for map algebra. Specifically, the Open Geospatial Consortium's proposal for handling a map (the coverage data type) lacks a set of functions to manipulate its content. To address this problem, our work proposes a specification for an algebra for Open GIS® coverages which uses a dimension-extended version of Egenhofer and Herring's 9-intersection predicates to express spatial operations. The proposed coverage algebra includes all Tomlin's functions, as well as operations that are not part of Tomlin's algebra, but are useful in practice. Our proposal could be the basis for setting up standard for operations on Open GIS coverages. The use of standards for operations in coverages would be a significant advance for increased interoperability of spatial data

UMA ÁLGEBRA PARA "*OPENGIS*® *COVERAGES* " BASEADA EM PREDICADOS TOPOLÓGICOS

RESUMO

Álgebra de mapas é uma coleção de funções para a manipulação de dados espaciais onde cada dado contém um conjunto de geometrias do mesmo tipo ligado a uma referência geográfica. A atual teoria para Álgebras de Mapas utiliza operadores *ad hoc* propostos por Dana Tomlin. Sua proposta obteve um grande sucesso prático, e a maioria das implementações de Sistemas de Informação Geográficos disponibiliza essas operações. Entretanto, existe uma carência de bases teóricas aos operadores propostos na álgebra de mapas de Tomlin. Já a proposta do Open Geospatial Consortium para a manipulação de mapas (ou coverages) necessita de um conjunto de funções para manipular esse conteúdo. Nosso trabalho propõe a especificação de uma Álgebra de Mapas para a coverage do Open Geospatial Consortium que utiliza uma versão extendida dos predicados topológicos propostos por Egenhofer e Herring. A álgebra para coverages proposta inclui todas as funções propostas por Tomlin, bem como operações que não são parte, mas que na prática são úteis. Nossa proposta pode ser uma base para o desenvolvimento de um padrão para as operações sobre Coverages.

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CHAPTER 1

INTRODUCTION

In recent years, there has been a significant effort to standardize the technology of geographical information systems (GIS). This effort is motivated by the large diffusion of GIS worldwide, the need to share geographical data, and for long-term maintenance of geographical archives. Sharing geographical data requires that different institutions, using diverse technologies, have access to the same data sets. Long-term archive maintenance needs that data outlives both its original media support and the software used to build it. Thus, sharing and maintenance of geographical data need standardization. The Open Geospatial Consortium (OGC) is developing standards for modelling, accessing, storing, and sharing geographical data. The extent of the effort involved in OGC's mandate is significant. Therefore, despite the progress already achieved, there are still areas where OGC's task is not complete. One of the missing parts of OGC's specification is a set of functions for manipulation of coverages, a subject commonly referred to as 'map algebra'. The current version of OGC's specification for coverages (Kottman, 2000) mentions unary and binary operations, and does not consider spatial operations. Since there is a literature on map operations ((Berry, 1987) (Frank, 1987) (Tomlin, 1990)), it is important to consider how these operations can be used for handling OGC's coverages. This work will discuss how to extend OGC's coverage data type to create an algebra for coverages.

The main contribution to map algebra comes from the work of Tomlin (Tomlin, 1990). Tomlin proposed a set of operations that has proven useful in practice. Extensions to Tomlin's map algebra include the GeoAlgebra of Takeyama and Couclelis (Takeyama; Couclelis, 1997) and MapScript, a language that includes dynamical models (Pullar, 2001). Extensions of map algebra for spatio-temporal data handling are discussed by Mennis et al. (Mennis *et al.*, 2005) and Frank (Frank, 2005). All these works use Tomlin's algebra as a basis for their work. The main problem with Tomlin's algebra and its extensions is their *ad hoc* basis. There is no foundation for

assessing the completeness of his algebra. Thus, there could be other types of operations that are missing in his algebra.

Therefore, one of the open challenges in spatial information science is to develop a theoretical foundation for a comprehensive set of operations on coverages. We need to find out if Tomlin's map algebra is part of a more general set. We state these questions as: *"What is the theoretical foundation for spatial operations on coverages?", "Could this theoretical foundation provide support for a comprehensive spatial algebra for coverages?"* To respond to these questions, we take the topological predicates of Egenhofer and Herring (1991) as a basis for defining an algebra for coverages. Using these predicates, we develop a coverage algebra that includes Tomlin's map algebra as a subset.

In what follows, we briefly review the literature about coverages, map algebra and spatial predicates in chapter 2. In chapter 3, we present an algebraic specification of an algebra for coverages. In chapter 4, we show the implementation of the algebra in the Haskell language, some examples of its use, including some problems that are not expressible in Tomlin's work and the implementation of it into a GUI Interface. To conclude, in chapter 5 we present our conclusions and pointing out future lines of research.

CHAPTER 2

LITERATURE REVIEW

In this chapter, we review the main basis for our work: OGC's definion of coverages, Tomlin's map algebra, and Egenhofer's topological operators.

2.1 The OpenGIS® Coverage

OGC's definition of coverage provides a support for the concepts of 'fields' and 'maps' (Kottman, 2000). A coverage in a spatial representation that *covers* a geographical area and divides it in spatial partitions that may be either regular or irregular and assigns a value to each partition. The computer representation of a coverage consists of a coverage function over a discrete domain, called the *DiscreteC_Function*. Its domain is a collection of geometries and its range is a set of vectors of attributes. The OGC specification describes the *DiscreteC_Function* as having four operations, shown in Figure 2.1:

- **Num**: finds the number of geometries in the coverage.
- **Values:** finds the possible values of the coverage function.
- **Domain:** finds the geometries in the domain of the coverage function.
- **Evaluate:** given a geometry, return a vector that represent the values of the coverage function for the associated location.

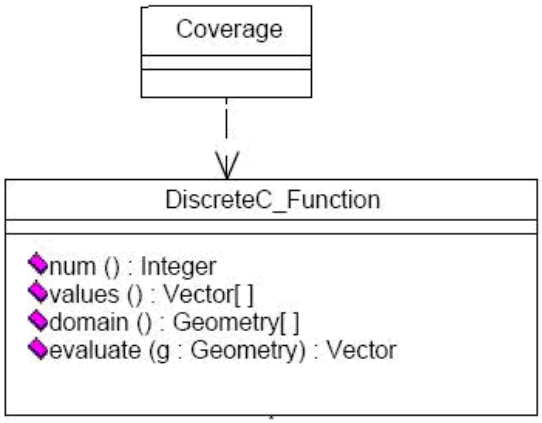


Figure 2.1 - The Open GIS discrete coverage function

Source: (Kottman, 2000)

An example is coverage whose domain is the geometries that describe the states of a country, and the range is each state's population in 2004. A second example would be a coverage whose domain is a set of regular cells and the range is a set of values representing the maximum and minimum yearly temperatures the region covered by the set of cells. OGC considers a set of coverage subtypes, where each subtype uses a different spatial data structure to build its domain. The subtypes include polygons, images, TINs, surfaces, and point sets. OGC describes how to evaluate the discrete coverage function for different coverage subtypes (Kottman, 2000).

2.1.1 The Discrete Spatial Domain

The discrete spatial domain of a DiscreteC_Function may be any geometry or collection of geometries.

The domain can be represented by a raster, i. e., a region composed by a planar-enforced set of closed homogeneous rectangles (named "pixels", "cells" or "locations"), like an image or a cell space. A coverage with this kind of domain maps from "locations" to values.

Furthermore, a domain can be composed of a collection of homogeneous geometries. A geometry can be represented by a point, line or region. A coverage with this kind of domain maps from geometries to values. Examples of elements in this

domain are rivers and roads, represented by lines, or states and forests, represented by polygons. The Figure 2.2 shows all the subtypes domains that a Coverage Function can assume.

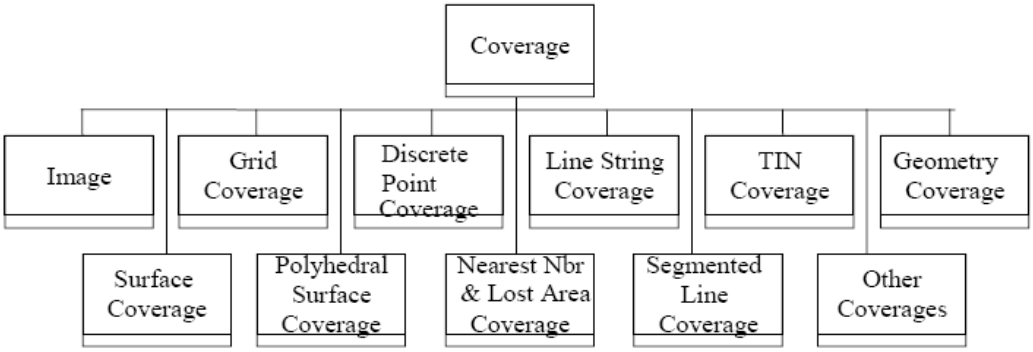


Figure 2.2 – Coverage Subtypes

Source: (Kottman, 2000)

2.1.2 The Range of a Coverage and the DiscreteC_Function

The range of a **DiscreteC_Function** is a set of vectors of attributes. For any geometry of the domain, the DiscreteC_Function maps to a vector of attributes.

DiscreteC_Function*: (geometry in spatial domain)* $\rightarrow (v_1, v_2, v_3, \ldots, v_n)$

The **DiscreteC_Function** can be seen like a set of functions with the same spatial domain, where each function maps to a unique attribute range.

$f_1 : p \rightarrow v_1, \ldots, f_n : p \rightarrow v_n$ where p is a geometry in the spatial domain.

For example, considering that the domain is composed by all states in a country, the **DiscreteC_Function** may assign to each state its population of the years 2004 and 2005, and the percentage of deforestation area. Besides, considering the domain the region of a forest portioned in a discrete way, a set of pixels, for each pixel the function may assign its temperature, elevation, dampness, and deforestation value.

2.2 Topological operators for spatial relations

The OGC specification describes a set of topological predicates for spatial relations between geometries of simple features (Herring, 2006). These predicates use a

dimension-extended version of the 9-intersection model proposed by Egenhofer and Herring (Egenhofer *et al.*, 1991). The 9-intersection model considers a geometrical object (A) as composed by a set of boundary points (∂A), a set of interior points (A^{o}), and a set of exterior points (A^{-}).

This model allows identifying 512 binary relations in $\Re^2$. For area-area relations the model presents a standard set of seven predicates {*'disjoint', 'equal', 'touch', 'inside', 'overlap', 'contains', 'intersects'*}, as adopted by the Open Geospatial Consortium (Figure 2.3). The predicate *'intersects'* is included for user convenience, following OGC's proposal (OGC, 1998). '*Intersects*' is defined as

intersects(a,b) ⇔ ! disjoint(a,b)

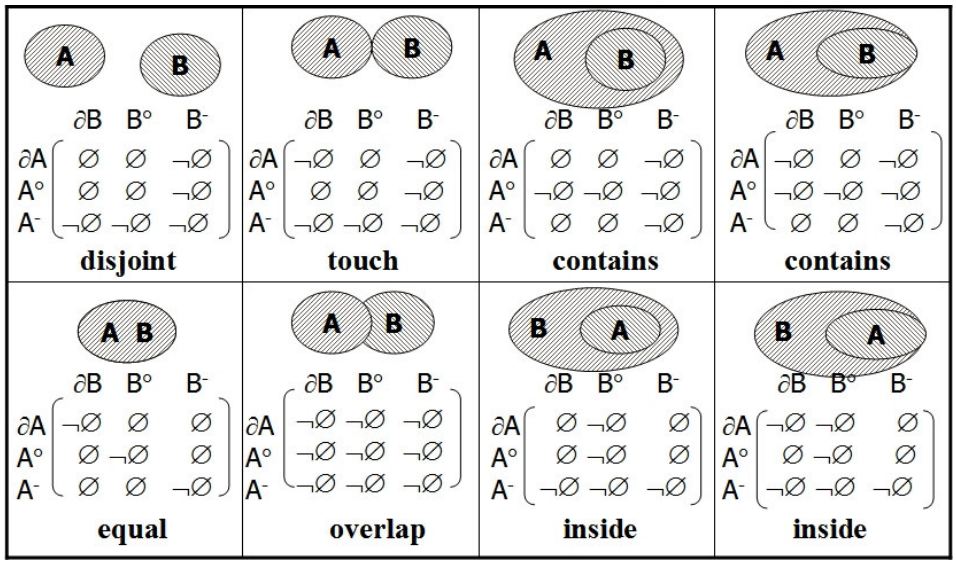


Figure 2.3 – Topological predicates for area-area based on the 9-intersection matrix. Adapted from Egenhofer and Herring (1991)

For line-area relations, the model proposes the set *{'disjoint', 'touch', 'within', 'cross', 'intersects'*}. For point-area relations, the model proposes the set *{'disjoint', 'touch', 'within', 'cross'*}. Other topological relations are described in (Egenhofer *et al.*, 1991).

The works of Winter (1995) and Winter and Frank (2000) extends the Egenhofer's 9-intersection model to the application to raster representations. Application of the Egenhofer's 9-intersection model is only defined over entities that have a set of interiors and exterior points. In a discrete raster representation of the geographical space its entities don't satisfy this requirement. Winter and Frank (2000) present a new hybrid raster model, where they defining a raster element with boundary, interior and exterior (Figure 2.4). So, "*In this hybrid raster representation, topological relations relate again to general topology in Euclidean space, and the four or nine-intersection can be applied in full accordance to vector representations."* Winter and Frank (2000).

Figure 2.4 – The Hybrid Raster Representation
Source: (Winter *et al.*, 2000)

In what follows, we define spatial operations in coverage algebra should using OGC's topological predicates.

2.3 Map Algebra

The main contribution to map algebra comes from the work of Tomlin. Tomlin's map algebra (1990) includes first-order and higher-order functions for maps. *First order* functions take values as arguments (these are the functions associated to the map values). *Higher order functions* are functions that have other functions as arguments. Higher order functions are the basis for map algebra operations (Frank, 1997). These functions apply a first-order function to all elements of map. Tomlin (Tomlin, 1990) proposes three higher-order functions

- **Local function**: produce a new map, whose value in each location *p* depends only of the values in *p* in the input maps, as in "*classify as unsuitable for farming all*

areas with slope greater than 15%". A local operation is a mapping between the *ranges* of the input and output fields (Figure 2.5).

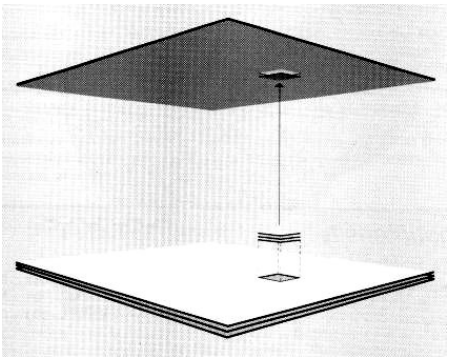


Figure 2.5 – Local Operation
Source: (Tomlin, 1990)

- **Focal function**: produce a new map, whose value in each location *p* depends only of the values of a neighbourhood around *p* in the input map, as in the expression "*for each county, calculate the average population of its neighbours*" (Figure 2.6). Focal functions use the condition of adjacency, which matches the spatial predicate *touch*.

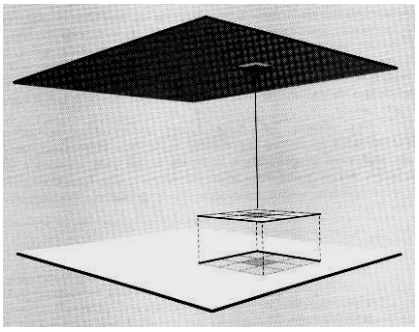


Figure 2.6 – Focal Operation
Source: (Tomlin, 1990)

- **Zonal function**: produce a new map, whose value in each location *p* depends on the values of a region in an input map. This region is defined by a restriction on a third map, called the reference map. Example is "*given a map of cities and a digital terrain model, calculate the mean altitude for each city"* (Figure 2.7). Zonal

functions use the condition of topological containment, which matches the spatial predicate *inside*.

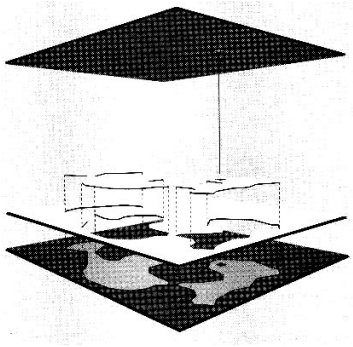


Figure 2.7 – Zonal Operation
Source: (Tomlin, 1990)

We can express Tomlin's spatial operations using Egenhofer and Herring's (Egenhofer *et al.*, 1991) topological predicates. The focal operation uses the condition of *adjacency*, which matches the spatial predicate '*touch'*. The zonal operation uses the condition of containment, which matches the spatial predicate '*within'*. Since *'touch'* and *'within'* are part of a more general set of predicates, Tomlin's operations use only a subset of all possible topological relations between areas. Tomlin's algebra uses spatial predicates in a limited way. It applies the *'touch'* relation (focal function) only over the same input map. It also only applies the *'within'* relation (a zonal operation) over a reference map. If we remove the limits of Tomlin's algebra, we can have a coverage algebra based on the full set of topological predicates, whose operations don't restrict the way in which the predicates are used.

CHAPTER 3

A GENERALIZED ALGEBRA FOR COVERAGES

This section presents the design of an algebra for coverages. The proposed algebra extends the *coverage* data type defined by the Open GIS® consortium (Kottman, 2000) and has *nonspatial* and *spatial* higher-order functions. The nonspatial operations are Tomlin's local operations. The spatial operations perform operations on coverages using topological predicates.

3.1 Conventions used in the text

We define the coverage operations using an algebraic specification of data types. In our definitions, we use CASL, the Common Algebraic Specification Language (Bidoit, 2004). CASL is a general-purpose language for both requirements and design specifications. A basic specification in CASL consists of a set of declarations of symbols, and a set of axioms and constraints, which restrict the interpretations of the declared symbols. For a detailed syntax for CASL specifications, see (Bidoit, 2004). We provide examples from the CASL user manual (Bidoit, 2004) to allow the reader to better follow our proposal. The first example has a unique sort and a predicate:

spec STRICT_PARTIAL_ORDER =
sort *Elem*
pred __ < __: *Elem* × *Elem*

axioms
forall x,y,z: Elem
· $x \leq x$ *%(reflexive)%*
· $x=y$ *if* $x \leq y \wedge y \leq x$ *%(antisymmetric)%*
· $x \leq z$ *if* $x \leq y \wedge y \leq z$ *%(transitive)%*

The STRICT_PARTIAL_ORDER specification uses a sort *Elem* and a binary infix predicate symbol '<'. Argument sorts are separated by the sign '×'. CASL uses '__ ' (pairs of underscores) as place-holders for arguments. The predicate '<' is associated to three axioms: reflectivity, antisymmetry and transitiveness. CASL provides the keyword **type** to shorten declarations of sorts and constructors, as in the example:

spec CONTAINER [**sort** *Elem*] =
type *Container*::= *empty* | *insert (Elem*: *Container)*
pred *__inside __: Elem* × *Container*
axioms
$\forall e, e' : Elem; C: Container$
- ¬(*e inside empty*)
- *e inside (insert (e', C))* $\Leftrightarrow$ *(e =* $e' \vee$ *e inside C)*

end

3.2 The Coverage Data Type

This section presents an algebraic specification for an extended version of Open GIS coverage. A coverage is a *c_function:: G→ V* over a finite collection of geometries G and a set of attribute values V. Without loss of generality, we will discuss the case of coverages where each geometry has only one value. The generalization to a coverage that returns a vector of values is simple, but would need a slightly more complicated notation. We will assume that *Geometry* and *Value* sorts are those used by OGC. The reader should refer to (Kottman, 2000) for details. We begin by defining a set of auxiliary specifications: *list* (a list of elements), *single and multiargument functions*, *comparison predicates* and *topological predicates*. For brevity's sake, we provide a limited list of single and multiargument functions and of selection predicates. We can extend these lists of functions if needed.

spec SINGLEFUNCTION =
sort *Value*
ops

log:	*Value → Value*
exp:	*Value → Value*

sin:	*Value* → *Value*
sqrt:	*Value* → *Value*

end

spec MULTIFUNCTION =
sort *Value*
ops

sum:	*Value* × *Value* → *Value*
product:	*Value* × *Value* → *Value*
maximum:	*Value* × *Value* → *Value*
mean:	*Value* × *Value* → *Value*
minimum:	*Value* × *Value* → *Value*

end

spec LIST [**sort** *Elem*]
type *List*::= *empty* | *cons (Elem*; *List*)
ops
length: List → *Integer*
end

spec COMPPRED =
sort *Value*
pred

__ **==** __ *:*	*Value* × *Value*
__ **<** __ *:*	*Value* × *Value*
__ **>** __ *:*	*Value* × *Value*
__ **!=** __ *:*	*Value* × *Value*

end

spec TOPOPRED =
sort *Geometry*
pred

__within__:	*Geometry* × *Geometry*
__overlap__:	*Geometry* × *Geometry*
__disjoint__:	*Geometry* × *Geometry*
__equal __:	*Geometry* × *Geometry*
__touch__:	*Geometry* × *Geometry*
__contains__:	*Geometry* × *Geometry*
__cross__:	*Geometry* × *Geometry*
__intersects__:	*Geometry* × *Geometry*

end

Using these definitions, we provide an abstract specification the *Coverage* data type. It uses the operations defined by OGC (see Figure 2.1 above) and includes three constructors, a new predicate and a new operation.

spec COVERAGE [**sort** *Geometry,* **sort** *Value,* **sort** *List*] =
type *Coverage::= empty* |
new (List [(Geometry, Value)])
subset (Coverage, List [Geometry])

ops

insert:	*Coverage × (Geometry, Value) → Coverage*
evaluate:	*Coverage × Geometry → Value*
domain:	*Coverage → List [Geometry]*
num:	*Coverage → Integer*
values:	*Coverage → List [Value]*
pred	
__contains__:	*Coverage × Geometry*

axioms

∀g, g': Geometry; *v: Value; C: Container*

- *contains (C, g) ⇔ g ∈ domain (C)*
- *evaluate (C, g) == error ⇔ contains (C, g) == false*
- *evaluate (insert (C, (g, v)), g) = v*
- *v = evaluate (C, g) ⇔ v ∈ values (C)*
- *num (C) = length (domain (C))*
- *values (C) = List [evaluate (C, g), ∀ g ∈ domain (C)]*
- *subset (C, List[g]) ⇔ (C₁ = empty ()) ∧ insert (C₁,(g, v)), ∀ g ∈ List [g] ∧ v = evaluate (C, g)*

end

The first constructor (*empty*) builds an empty *Coverage*. The second constructor (*new*) builds a new *Coverage* by providing a list of *(Geometry, Value)* pairs. The third constructor (*subset*) builds a new *Coverage* by extracting a subset of the original locations. The predicate *contains* verifies if a certain instance of *Geometry* is in the *Coverage*. *Insert* includes a new *(geometry, value)* pair in the coverage. *Evaluate* takes a coverage and a geometry and produces an output value ("*give me the value of the coverage at location g*"). *Domain* returns the geometries of the coverage's domain. *Num* returns the number of geometries on the coverage's domain. *Values* returns the list of values of the coverage's range. The axioms point to the various restrictions on the *Coverage* specification. The last axiom shows that creating a *Coverage* from a subset of locations of an existing one is the same as building an empty *Coverage* and then inserting a list of *(Geometry, Value)* pairs.

3.3 Coverage operations

Operations on coverages are operations that produce a new coverage, and include nonspatial and spatial ones. For *nonspatial operations*, the value of a location in the output map depends on the values of the same location in one or more input maps. They

include logical expressions such as "*classify as high-risk all areas without vegetation with slope greater than 15%" and "find the average of deforestation in the last two years"*. We consider three types of nonspatial operations: *single* (single argument operations), *multiple* (multiple argument operations) and *select* (nonspatial selection using a comparison predicate).

Spatial operations are higher-order functions that use a topological predicate and generalize Tomlin's focal and zonal operations. They include expressions such as *"calculate the local mean of the map values"* and *"given a map of cities and a digital terrain model, calculate the mean altitude for each city"*. These operations take two coverages (the *reference coverage* and the *input coverage*) and produce a new coverage, which has the same geometries as the reference coverage. A spatial operation has two parts: *spatial query* and *composition*. The *spatial query* operation starts by finding the matching geometry in the *reference* coverage for each location p in the new coverage. Then it applies a topological predicate between that geometry and all geometries of the *input* coverage. The result of the spatial query is a new (temporary) coverage containing the input geometries that match the predicate. The *composition operation* then applies a function to the values on the new coverage to produce the result (Figure 3.1).

Take the expression *"given a coverage of cities and a coverage of altitudes, calculate the mean altitude for each city"*. In this expression, the *input coverage* is the altitude one and the *reference coverage* is one with cities. To evaluate the expression, we first select the terrain values within each city. This uses the selection operator with the *within* predicate. Then, we calculate the average of each set of these values. This uses the composition operator with the *mean* function.

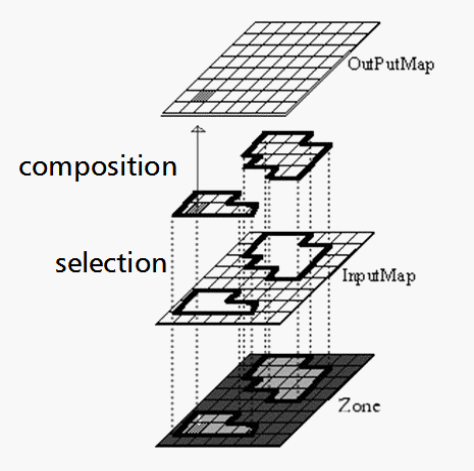


Figure 3.1 – Spatial Operations (selection + composition).
Adapted from (Tomlin, 1990)

To provide the abstract specification for the coverage operations, we distinguish between nonspatial and spatial operations. For spatial operations, we define two auxiliary functions (*query* and *compose*). We use the CASL the **'local … within'** construct for such needs. This construct distinguishes between operations which are visible outside the specification and auxiliary functions.

spec COVERAGE_OPERATIONS

[**sort** *Geometry*, **sort** *Value*, sort *Coverage*, **sort** *SingleFunction*, sort *MultiFunction,* **sort** *List,* **sort** *CompPred* **sort** *TopoPred*]

ops

single:	$SingleFunction \times Coverage \rightarrow Coverage$
multiple:	$MultiFunction \times List[Coverage] \rightarrow Coverage$
select:	$Coverage \times CompPred \times Value \rightarrow Coverage$

local ops

sp_query:	$Coverage \times TopoPred \times Geometry \rightarrow Coverage$
compose :	$MultiFunction \times Coverage \rightarrow Integer$

within op

spatial:	$MultiFunction \times Coverage \times TopoPred \times Coverage \rightarrow Coverage$

axioms

$\forall g, g_1$: *Geometry*; v, v_1: *Value*; C, C_1, C_2: *Container*, *topo: TopoPred, comp: CompPred, fs: SingleFunction, fm: MultiFunction*

- *single (fs, C_1) = new (List[g, v]),*
  $\forall (g, v) \mid g \in domain(C) \wedge v = fs(evaluate(C, g))$
- *multiple (fm, C_1, C_2, ..., C_n) = new (List[g, v]),*
  $\forall (g, v) \mid g \in (domain(C_1) \cap domain(C_2) \ldots) \cap domain(C_n)) \wedge$
  $v = fm(evaluate(C_1, g), evaluate(C_2, g)))$
- *select (C, comp, v_1) = subset (C, S), where*
  $S = List[g \mid g \in domain(C) \wedge comp(evaluate(C, g), v_1)]$
- *sp_query (C_1, topo, g_1) = subset (C, S), where*
  $S = List[g \mid g \in domain(C) \wedge topo(g, g_1)]$
- *compose (fm, C) = fm (values (C))*
- *spatial (f, C_1, topo, C_2) = new (List[g, v]),*
  $\forall (g,v) \mid g \in domain(C_2) \wedge v = compose(fm, sp_query(C_1, topo, g))$

end

The first axiom describes the *single* operation, which applies a function to all values of the input. The second axiom describes the *multiple* operation, which applies a multivalued function to all values of the input. The third axiom describes the nonspatial selection operation. The fourth axiom shows that the *spatial query* selects a subset of the original coverage whose geometries satisfy a topological predicate ("*select all deforested areas within the state of Amazonas*"). The fifth axiom describes the *composition* operation. The last axiom describes the *spatial* function. It builds a new coverage by providing a list of *(Geometry, Value)* pairs. The geometries of the *new coverage* are the same as those of the *reference coverage*. We get the values of the new coverage by combining a *spatial query* and a *composition*. In the next section, we show how these operations for coverages are enough for a comprehensive algebra.

3.4 Expressiveness of Topological Operators for Coverage Algebra

All coverage subtypes proposed by OGC are implemented as discrete geometrical spatial data structures. Thus it is important to consider what extent the topological operators can cover the spatial relations between these coverage subtypes. As discussed above, spatial operations have two parts: a spatial query and a composition on the values selected by the query. The expressive power of the spatial query is limited by the capacity of computing them it in each spatial data structures. Based on OGC's proposal

for coverage subtypes (Kottman, 2000), we can distinguish different types of discrete data structures for coverages:

- *2,5D structures:* TIN coverages.
- *2D Area-based structures:* grid coverages, images, polygon coverages, surfaces.
- *1D Line-based structures*: segmented line coverages, line string coverages.
- *0D point-based structures*: discrete point coverages, nearest neighbour coverages, lost area interpolation.

Since the dimension-extended version of the 9-intersection model only handles 2D, 1D and 0D data structures, the proposed coverage algebra cannot be used for TIN coverages. Also, consider that a spatial query operation involves two coverages: the reference and the input coverages (see Section 3.2). Thus, application of topological operators depends on the geometries of the reference and the input coverages, as outlined in Table 3.1.

Table 3.1 – Topological operators applicable to spatial operations on coverages

Input coverage type	**Reference coverage type**	**Operators**
area	area	*{'disjoint', 'equal', 'touch', 'within', 'overlap', 'contains', 'intersects'}*
area	line	*{'disjoint', 'touch', 'intersects', 'contains'}*
area	point	*{'disjoint', 'touch', 'contains'}*
line	area	*{'disjoint', 'cross', 'touch', 'within', 'intersects'}*
line	line	*{'disjoint', 'equal', 'touch', 'within', 'overlap', 'intersects'}*
line	point	*{'disjoint', 'touch', 'contains'}*
point	area	*{'disjoint', 'touch', 'within'}*
point	line	*{'disjoint', 'touch', 'within'}*
point	point	*{'disjoint', 'equal'}*

3.5 Examples of Coverage Algebra

To provide examples of the proposed coverage algebra, we propose a shorthand notation for the operations, as follows:

Table 3.2 – Convenience shorthand for non-spatial operators

```
new_cov:= singlefun in_cov;   % single value functions%
new_cov:= multifun [in_cov]; % multivalued functions%
new_cov:= in_cov comp_pred value; % selection%
new_cov:= multifun in_cov topo_pred ref_cov;

%for spatial operations%
```

The parameters for the operations are:

- `new_cov` is the coverage with the new values.
- `singlefun` is a single argument function as given in section 3.2.
- `multifun` is a multiargument function as given in section 3.2.
- `in_cov` is the input coverage.
- `[in_cov]` is a list of input coverages.
- `comp_pred` is a comparison predicate.
- `value` is the comparison value.
- `topo_pred` is a topological predicate *(section 3.2)*
- `ref_cov` is the reference coverage used as a basis for applying the topological predicate.

The examples of the Table 3.3 show the use of non-spatial operators applied to coverages Table 3.4 shows examples of spatial operations.

Table 3.3 – Examples of non-spatial operators

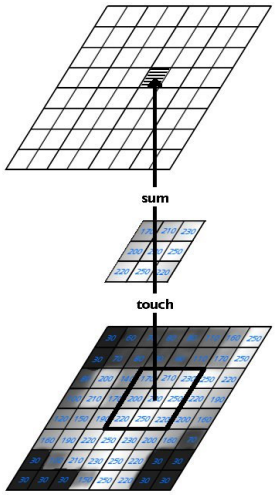
Informal description	***Coverage Algebra Expression***
"Find the square root of the topography"	`topoSqrt:= sqrt topography;`
"Find the square root of the cities' population"	`popSqrt := sqrt  cityPop;`
"Find the average of deforestation in the last two years"	`defAve := mean (defor2004, defor2003);`
"Select areas higher than 500 meters"	`highM := topography > 500;`

"Select the cities with the population higher than 50.000"	`highPop := cityPop > 50000;`

Table 3.4 – Examples of Spatial Operations

Informal description	***Coverage Algebra Expression***
"Given a coverage of cities and one of states, find the total population for each state"	`statePop := sum cityPop within statePop;`
"For each cell, calculate the average deforestation of its neighbours"	`fsum := sum defor touch fsum;`
"Given a coverage of rivers and one of cities with population, find the number of people that live along each river".	`riversPop:= sum cityPop intersects rivers;`
"Given a coverage of cities and one of deforestation, find the average deforestation of each city".	`ave_city_defor:= average defor within city;`
"Given a coverage containing roads and a one of deforestation, calculate the mean deforestation along each road".	`defRoads:= mean defor intersects roads;`

Consider the operation: *"For each cell, calculate the average deforestation of its neighbours*". In Tomlin's algebra, this is a focal operation (Figure4.b). Considering the deforestation map (`defor`) as input and the local sum map (`lsum`) as output, we state the operation as shown in Figure 5.



`fsum := sum defor touch fsum;`

Figure 3.2 – Focal sum of deforestation

Note one interesting feature: the result (`fsum`) is also the reference coverage for the spatial predicate `(defor touch fsum)`. This syntax may seem odd at first sight, but it follows from the generality of the proposal. By taking the reference map and the new map to be the same, we ensure the outcome satisfies the condition ("local mean"). Had we used a third map as a reference, the result would be different if this map would not have the same spatial partitioning as the output map.

CHAPTER 4

PROGRAM ENVIROMENT

This chapter shows how the proposal map algebra was implemented using the Haskell functional language and integrated with a spatial database using TerraLib. A graphical interface was build to help the users to write and maintains his programs. We also show some examples of it use and compare our proposal to Tomlin's map algebra.

4.1 TerraHS

To make our algebra useful was necessary to implement it in a computer language. The implementation was made in the Haskell functional language (Jones, 2002), (Peyton Jones *et al.*, 1999) and (Thompson, 1999). Functional programming is a programming paradigm that considers that computing is evaluating of mathematical functions. Functional programming languages are convenient to translate algebraic specifications into testable code (Frank e Kuhn, 1995; Frank e Medak, 1997) and functional languages express the semantics of abstract data types directly, an essential property for formal specification languages (Frank e Kuhn, 1995). The Haskell report describes the Haskell language as:

> *"Haskell is a purely functional programming language incorporating many recent innovations in programming language design. Haskell provides higher-order functions, non-strict semantics, static polymorphic typing, user-defined algebraic datatypes, pattern-matching, list comprehensions, a module system, a monadic I/O system, and a rich set of primitive datatypes, including lists, arrays, arbitrary and fixed precision integers, and floating-point numbers*" (Jones, 2002).

TerraLib is a spatial database programming environment that provides support to storage and retrieve spatial data (Câmara *et al.*). TerraHS (Costa, 2006) was created as an application that enables developing geographical applications in the Haskell functional language using data stored in a spatial database manipulated with TerraLib.

4.2 Haskell Implementation

The Haskell implementation of the algebra was presented in (Costa, 2006). In this section we will show this implementation, concentrating in how to translate the data storage in a TerraLib database to coverages and in the syntax language that a user will use to write his programs.

4.2.1 Data Type

Based on the Coverage specification on section 3.2, in TerraHS was defined the *type class* Coverages in Haskell, described bellow:

```
class Coverages cov where
 evaluate :: (Eq a, Eq b) => cov a b → a → Maybe b
 domain :: cov a b → [a]
 num :: cov a b → Int
 values :: cov a b → [b]
 new_cov :: [a] → (a → b ) → (cov a b)
 fun :: (cov a b) → (a → b)
```

The functions of type class Coverages are parameterized on the input type a and the output type b.

- `evaluate` is a function that takes a `coverage` and an input value `a` and produces an output value.
- `domain` is a function that takes a `coverage` and returns the values of its domain.
- `num` returns the number of elements of the `coverage`'s domain.
- `values` returns the values of the `coverage`'s range.
- `new_cov`, a function that returns a new `coverage`, given a domain and a *coverage function.*
- `fun:` given a `coverage` , returns its *coverage function*.

The instance of the type class Coverages to the Coverage data type is shown below:

```
instance Coverages Coverage where
new_cov a f = (Coverage (f, a))
evaluate f o
    | (elem o (domain f)) = Just ((fun f) o)
    | otherwise = Nothing
domain (Coverage (f, a)) = a
num f = length (domain f)
values f = map (fun f) (domain f)
fun (Coverage (f,_)) = f
```

TerraHS provides support to the object-set data type storage in TerraLib. Object-set in TerraLib database is represented by a layer of geo-objects in a vector representation and a table of attributes. Where each object of the layer is linked with one line of the table. When we map an object set to a Coverage Function, we can take the layer as the domain of the function and the range of one column of attribute as the range of range of the function. The mapping of the function is given by the relationship of the object and its attribute. The Figure 4.1 shows an example of retrieving and storing a coverage

```
-- db connection
host = "localhost";
dbname =  "tedbteste";
user = "root";
password = "";
db = database ( dbname, host, user, password, MySQL )

-- layers and attributes
layer_def = "def_cel";
ldef = layer (db, layer_def)
att_def  = attr (ldef , "luc_def")

layer_def_out = "def_cel_out";
ldef_out = layer (db, layer_def_out)

main:: IO ()
main = do
     cov_def <- loadMap att_def
     let cov_def_sqrt = (single fsqrt cov_def)
     saveMap cov_def_class "FSQRT" ldef_out
```

Figure 4.1 - Retrieving and storing a coverage

As showed in the example, to retrieve and storage the information from a TerraLib database we need to inform the host, dbname, user name and password to make the connection with the database.

The *loadMap* function connects to the database, loads the geo-object set, converts these geo-objects into a coverage, and return it as output. The *saveMap* function converts a coverage to a geo-object set that will be saved in the database. If the column of attribute name not exist in the layer, it will be create.

4.2.2 Operations

We use a generic type class for the map algebra operations, as follows:

```
class (Coverages cov) => CoverageOps cov where
   single :: (b → c) → (cov a b) → (cov a c)
   multiple :: ([b]→c) → [(cov a b)] → (cov a b)→(cov a c)
   select :: (cov a b) → (a → c → Bool) → c → (cov a b)
   compose :: ([b] → b) → (cov a b) → b
   spatial :: ([b] → b) → (cov a b) → (a → c → Bool)
              → (cov c b) → (cov c b)
```

The implicit assumption of these is that the geographical area of the *output map* is the same of the *reference map*.

The instantiation of the coverage operations is provided by:

```
instance CoverageOps Coverage where
-- non-spatial operation on a single coverages
single f1 c = new_cov (domain c) ( f1 . (fun c))

-- non-spatial operation on multiple coverages
multiple fn clist c = new_cov (domain c)
(\x -> fn (faux clist x))

-- spatial selection
select cov pr o = new_cov sel_dom (fun cov)
where sel_dom = [l | l <- (domain cov) , (pr l o)]

-- spatial composition
compose f cov = (f (values cov))

-- spatial operation : selection + composition
spatial fn c1 predic cref = new_cov (domain cref)
```

```
(\x -> compose fn (select c1 predic x))
```

The *single* function has two input arguments: a *coverage (input_cov)* and a first-order function (f1), and produce a new *coverage* as output (new_cov). Its general syntax is:

```
let new_cov = single f1 input_cov
```

The word "**let**" is a keyword of the language, and the word " **single**" is used to specify the kind of operation.

The *multiple* function has three input arguments: a multivalued function (fn), a coverage list, and a reference coverage (ref_cov), and produces a new *coverage* as output. Its general syntax is**:**

```
let new_cov = multiple fn [input_cov1, input_cov2] ref_cov
```

The *spatial selection* has three input arguments: an input *coverage*, a predicate (predicate) and a reference element (geo), and generates a new *coverage* as output. Its general syntax is**:**

```
let new_cov = select input_cov predicate geo
```

The *composition function* has two input arguments, an input coverage and a multiple arguments function and generates a single value as output. Its general syntax is**:**

```
compose fn input_cov
```

The spatial function has three input arguments, a reference coverage, a multivalued function and a select predicate, generating a new coverage as output. Its general syntax is:

```
let new_cov = spatial fn input_cov predicate ref_cov
```

4.3 Integration into GUI software

A graphical interface was developed to the create and edit of the TerraHS programs. This graphical interface was integrated with TerraView (Câmara *et al.*) as a plugin.

4.3.1 Main Window

The main window has a text box that allows the edition of the programs (Figure 4.2). When the user creates a new program, the text starts with the includes necessaries to compile and run the Haskell program, and also sets the variables necessaries to the database connection, according to the database connection opened in the TerraView.

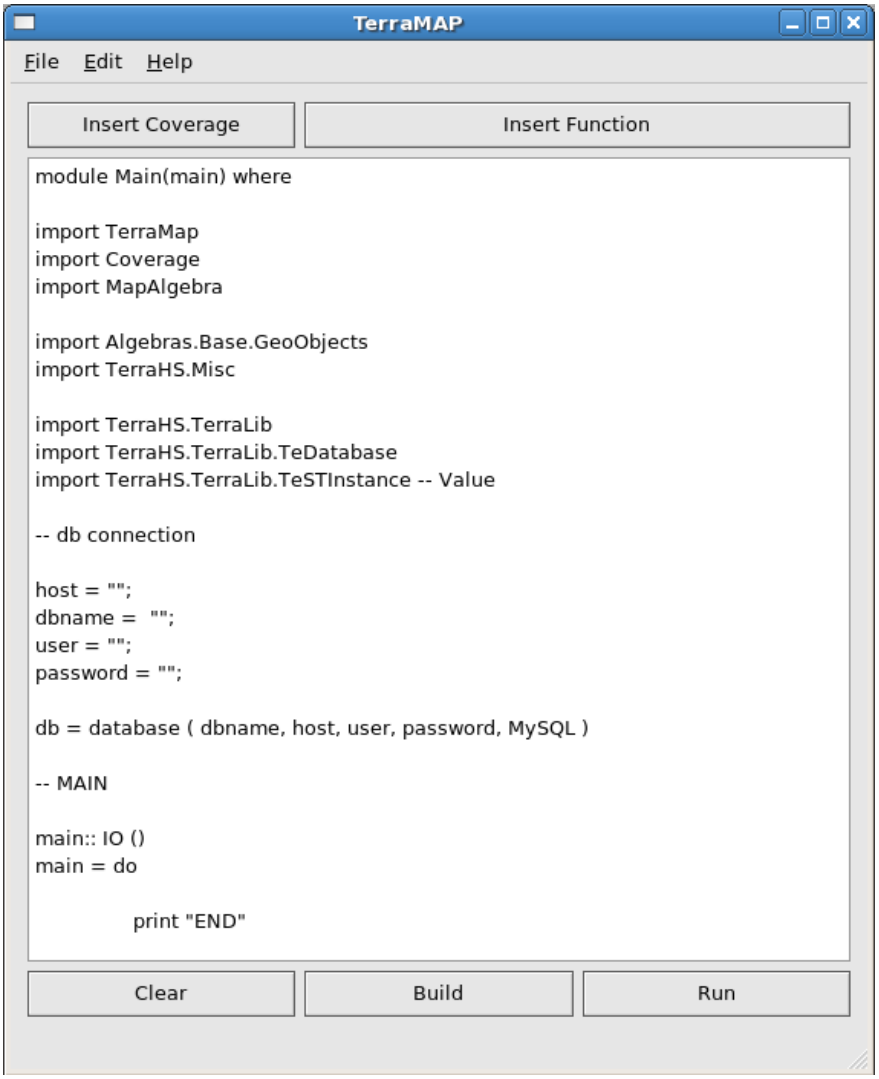


Figure 4.2 – The main window

The interface presents two buttons relatives to the program generation; these buttons assist the user to insert a coverage and to insert a operation in the program,

respectively Insert Coverage and Insert Operation. We also place three buttons to compile and run a program. The Clear button is used to delete the intermediate files generated by previous compilation of the program. The Build button is used to compile the program and to generate an executable file. The RUN button executes the program.

4.3.2 Algebra Interfaces

The sort of windows showed above are used to Insert a Coverage, Insert a Single Argument Function, Insert a Multiple Argument Function and Insert a Spatial Function. When we conclude the procedure of insertion, it generates a text of the operation and insert this text in the text box of the Main Window

- The window of the Figure 4.3 is used to insert a Coverage. To insert a coverage we select the layer and the attribute; after this we give a name to the coverage.

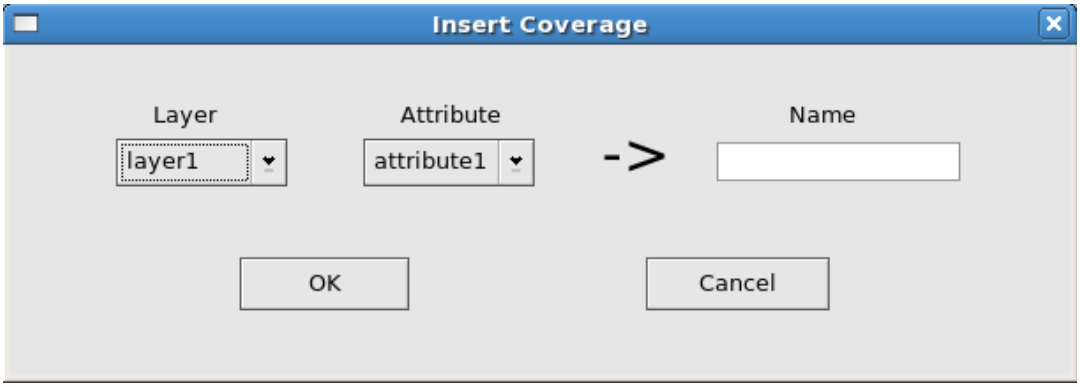


Figure 4.3 – Insert Coverage

- To insert a single argument function we need to select the output coverage, the single argument function and the input map (Figure 4.4).

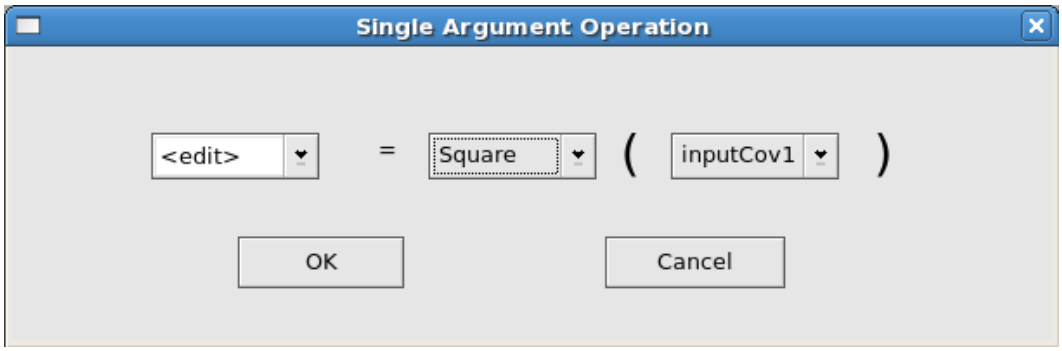


Figure 4.4 - Insert Single Argument Function

- To insert a multiple argument function we need to select the output coverage, the multiple argument function and the list of input coverages (Figure 4.5).

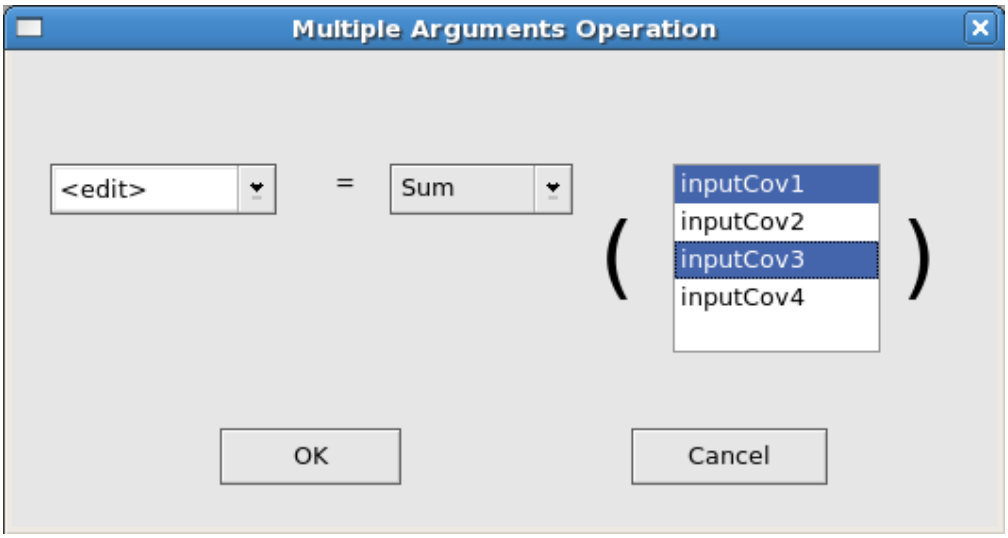


Figure 4.5 - Insert Multiple Arguments Function

- To insert a spatial function we need to select the output coverage, the multiple argument function, the input coverage, the spatial predicate and the reference coverage (Figure 4.6).

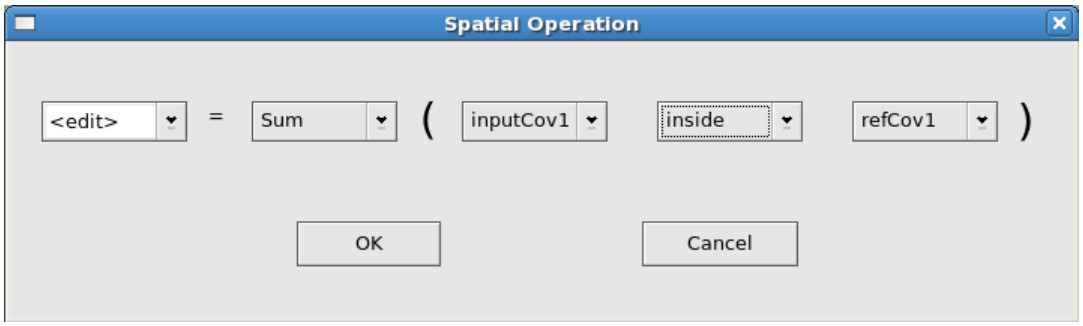
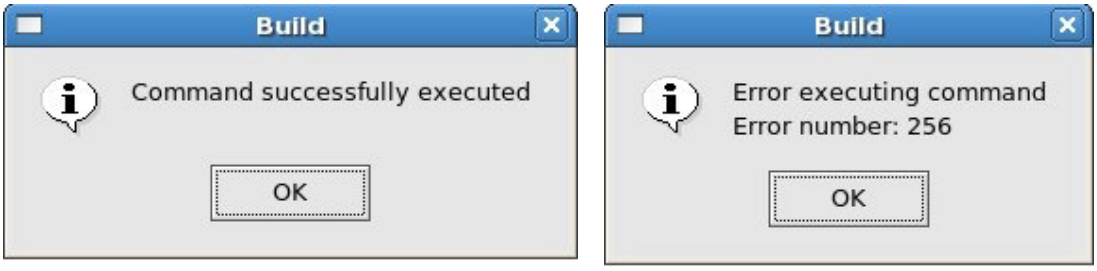


Figure 4.6 – Insert Spatial Function

- When the program is ready we compile it, pressing the "Build Button". If the program is correct the Message Box of the Figure 4.7.a will appear, else *Message Box* of the figure Figure 4.7.b will appear.



a) b)

Figure 4.7 – Build Windows

- If the program was build with success, to run it we need to press the "Run button". If everything is correct the *Message Box* of the figure Figure 4.8.a) will appear, else *Message Box* of the figure Figure 4.8.b) will appear.

a) b)

Figure 4.8 – Run Windows

4.4 Examples

The examples use INPE's database of deforestation of the Brazilian Amazonia (Aguiar, 2006), We selected a data set from the central area of the Pará state, composed of three coverages: deforestation (grid cells of 25 x 25 km^2), roads (lines), and protected areas (polygons), as shown in Figure 4.9.

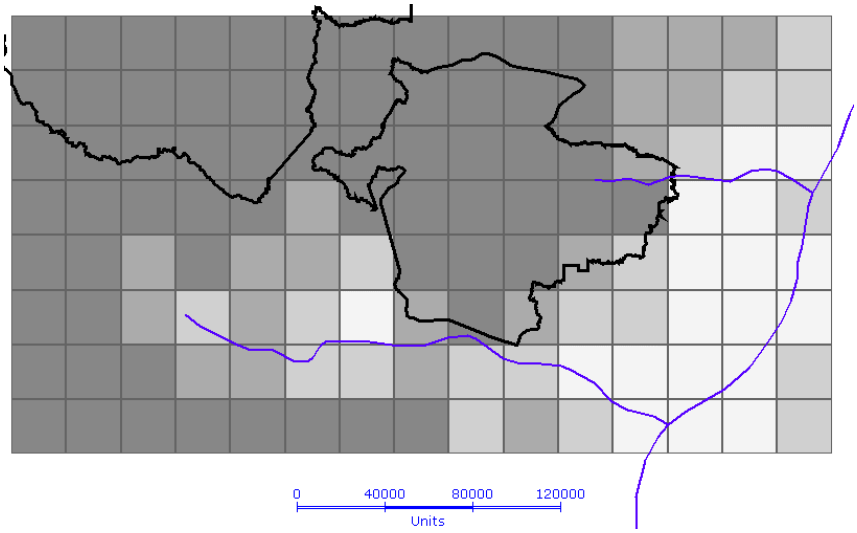


Figure 4.9 – Deforestation, protected areas and roads (Pará State, Brazil)

The Figure 4.10 show the main header of the program used for all the programs of the examples below.

```
module Main(main) where

--import
…

-- db connection

host = "localhost";
dbname =  "tedbteste";
user = "root";
password = "";

db = database ( dbname, host, user, password, MySQL )

-- layers and attributes

layer_def = "def_cel"
ldef = layer (db, layer_def)
att_def  = attr (ldef , "luc_def")

layer_roads = "roads_lin";
lroads = layer (db, layer_roads)
att_roads = attr (lroads, "object_id_3")

layer_pa = "pa_pol";
lpa = layer (db, layer_pa)
att_pa = attr (lpa, "object_id_1")

-- MAIN

main:: IO ()
main = do

     cov_def <- loadMap att_def
     cov_roads <- loadMap att_roads
     cov_pa <- loadMap att_pa
```

Figure 4.10 - Main header of the TerraHS programs

We instance the three main coverages of the example:

- `cov_def`: the deforestation coverage
- `cov_roads`: the roads
- `cov_ap`: the protection areas

Our first example considers the expression: "*Given a coverage of deforestation and a classification function, return the classified coverage*". The input is the deforestation coverage and the output is a classified coverage (`cov_def_class`) The classification function defines four classes: (1) dense forest; (2) mixed forest with agriculture; (3) agriculture with forest fragments; (4) agricultural area. This function is:

```
classify :: Value → Value
classify v
     | v < 0.2                    = "1"
     | ((v > 0.2) && (v < 0.5))   = "2"
     | (v > 0.5) && (v < 0.8)     = "3"
     | v > 0.8                    = "4"
```

We get the classified map (Figure 4.11) using the expression

```
let cov_def_class = single classify cov_def
```

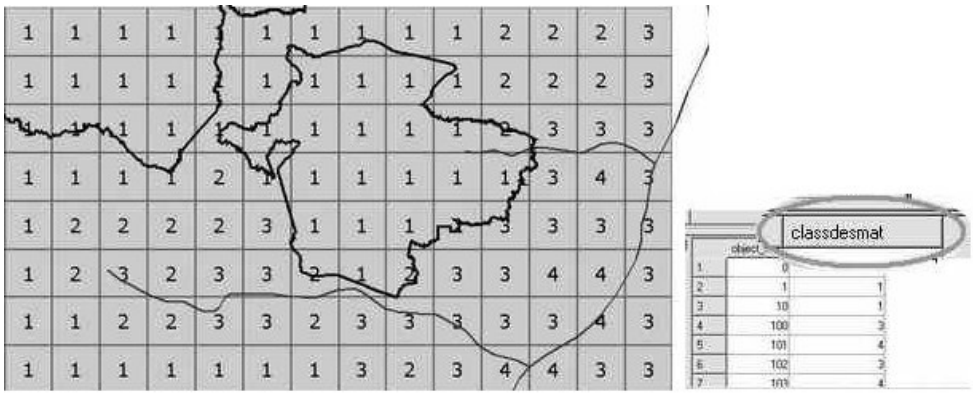


Figure 4.11 – Resulting coverage with classified deforestation

As a second example, take the expression: *"Calculate the mean deforestation for each protection area"*. The inputs are: the deforestation coverage (`cov_def`), a spatial predicate (`within`), a multivalued function (`mean`) and the coverage of protected areas (`cov_pa`). The output is a coverage of the protected areas (`cov_def_prot`) with the same objects as the reference coverage (`cov_pa`) and the deforestation for each area.

```
let cov_def_prot = spatial mean cov_def within cov_ap
```

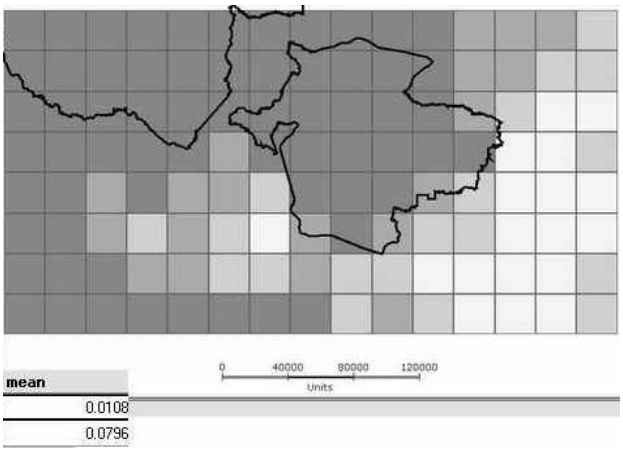


Figure 4.12 – Deforestation mean by protection area

Our third example is the expression: "*Given a coverage containing roads and one with deforestation, calculate the mean of the deforestation along each roads*". We have as inputs: the deforestation coverage (`cov_def`), a spatial predicate (`intersect`), a multivalued function (`mean`) and a road map (cov_`roads`). The product is a roads coverage with one value for each road. This value is the mean of the cells that intercept this road.

```
let cov_def_roads = spatial mean cov_def interest cov_roads
```

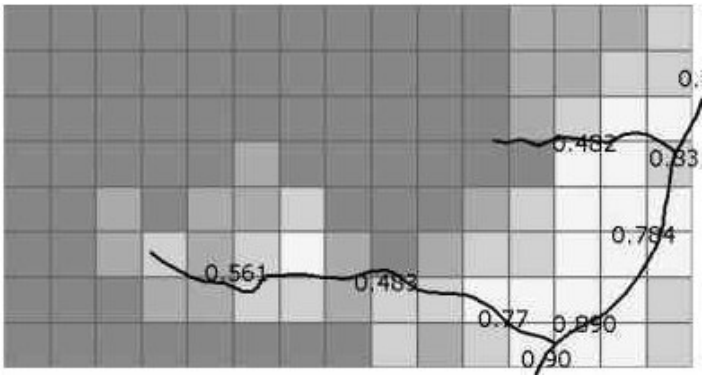


Figure 4.13 – Deforestation mean along the roads

Table 4.1 presents a comparison between the spatial operators as expressed in our proposal and in Tomlin's map algebra. The examples show that the proposed map

algebra expresses the focal and zonal functions of Tomlin's map algebra, using the 'touch' and *'within'* topological predicates. Operations involving '*overlap*', 'contains' and '*intersects*' predicates are part of our proposed coverage algebra and are not directly expressible by Tomlin's algebra. This shows that the proposed algebra is richer than Tomlin's, as well as having a solid conceptual basis.

Table 4.1 – Comparison of spatial operators with Tomlin's map algebra

Informal Description	*Generalized Map Algebra*	*Tomlin*
"Focal mean of topography"	`fmean:= mean topo` `touch fmean`	`FOCALMEAN OF TOPOGRAPHY`
"Given a coverage of cities and one of topography, find the mean altitude for each city."	`altcit:= mean topo` `within city`	`ZONALMEAN OF TOPOGRAPHY WITHIN CITIES`
"Given a coverage of national forests, get the deforestation at the edges of each forest"	`defBord:= sum defor` `overlaps forests`	**(no equivalent)**
"Calculate the mean of the deforestation along the road"	`defRoad:= mean defor` `intersects road`	**(no equivalent)**

CHAPTER 5

CONCLUSION AND FUTURE WORKS

Map algebra is a fundamental class of operations for spatial data sets. Most of the current implementations of map algebra use Tomlin's (Tomlin) proposal for local, focal and zonal operations. However, Tomlin's proposal uses *ad hoc* concepts and lacks a sound theoretical basis. This work addresses this problem, by proposing a new foundation for operations involving coverages. We have designed a coverage algebra that uses topological predicates to express spatial operations and that includes Tomlin's algebra as a subset.

There is one important set of operations on coverages that is not part of our proposal nor of Tomlin's: *convolution operators.* A convolution operation requires two coverages C_1 and C_2 produces a third coverage C_3. The value of each point p of C_3 is the integral of the product of C_1 and C_2, when C_2 is shifted so that its central point is coincident with p. From a conceptual point of view, convolutions are not part of map algebra, since the geometrical support for the second coverage C_2 (also called a *mask*) changes for each point of the output coverage. Convolution does not involve topological relations, but rather the definition of an integral function.

Our proposal points to a situation where all modeling of topological relations in two-dimensional spatial datasets can be handled by the 9-intersection model (dimension-extended), both for simple features and for coverages. Spatial data sets of higher dimensions (e.g., TIN coverages) need a different foundation. The foundation for handling spatial relation of higher dimensions requires topological operators that operate on 3D surfaces (Egenhofer, 2005). Convolution operations are a special case and need to be handled separately. A possible extension to our algebra would be to consider directional relations (Frank, 1992), which would be useful to express operations such as *"find the population of all cities north of the river"*.

The development of the coverage algebra application using TerraHS in the Haskell Language enabled it faster, easy and precise implementation. The use of this implementation in real problems demonstrate the usability of the algebra. Is desirable that in the future the implementation of coverage algebra can manipulate all the kind of geographical data available at TerraLib.

Another possible future work is to extend our algebra to the spatiotemporal domain. Examples of the proposal spatiotemporal algebras are presented in (Güting; Schneider, 2005), which defines an algebra for moving objects and (Medak, 2001), which proposes an algebra for modeling change in socio-economical units. To do this we need to extend the geographical elements of the overage to spatiotemporal domain, extend our algebra operations and extend the topological predicates to include temporal predicates.

Even considering the limitations, the expressiveness of the proposed coverage algebra is considerable. Given that it is based on a solid foundation, it could be considered as the basis for setting up standard for operations on Open GIS coverages. The use of standards for operations in coverages would be a significant advance for increased interoperability of spatial data.

CHAPTER 6

REFERENCES

AGUIAR, A. P. D. D. **Modeling Land Use Change in the Brazilian Aamazon: Exploring Intra-Regional Heterogeneity**. (Doctor Thesis). Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 2006.

Berry, J. K. Fundamental Operations in Computer-Assisted Map Analysis. **International Journal of Geographical Information Systems**, v. 1, n. 2, p. 119–136

Bidoit, M., Mosses, P.D.: CASL User Manual. Lecture Notes in Computer Science 2900 (IFIP Series). Springer, Heidelberg (2004)

Câmara, G.; Souza, R.; Pedrosa, B.; Vinhas, L.; Monteiro, A. M.; Paiva, J.; Carvalho, M. T.; Gattass, M. TerraLib: Technology in Support of GIS Innovation. In: II Brazilian Symposium on Geoinformatics, GeoInfo2000, 2000, São Paulo.

Costa, S., Camara, G., Palomo, D.: TerraHS: Integration of Functional Programming and Spatial Databases for GIS Application Development. In: Davis, C. (ed.): VIII Brazilian Symposium in Geoinformatics, GeoInfo 2006. SBC, Campos do Jordão, Brazil (2006)

Egenhofer, M.: Spherical Topological Relations. Journal on Data Semantics **III** (2005) 25-49

Egenhofer, M.; Herring, J. **Categorizing Binary Topological Relationships Between Regions, Lines, and Points in Geographic Databases**. Orono, ME: Department of Surveying Engineering, University of Maine, 1991.

Frank, A. Overlay Processing in Spatial Information Systems. In: Chrisman, N. R. (Ed.). **AUTO-CARTO 8, Eighth International Symposium on Computer-Assisted Cartography**. Baltimore, MD, 1987, p. 16-31.

Herring, J.: OpenGIS® Implementation Specification for Geographic information - Simple feature access - Part 1: Common architecture (version 1.2.0). Open Geospatial Consortium, Wayland, MA (2006)

Kottman, C., Roswell, C.: The OpenGIS Abstract Specification Topic 6: The Coverage Type and Its Subtypes (Version 4). Open Geospatial Consortium, Wayland, MA (2000)

___. Qualitative Spatial Reasoning about Distances and Directions in Geographic Space. **Journal of Visual Languages and Computing**, v. 3, n. 4, p. 343-371

___. Higher order functions necessary for spatial theory development. In: Auto-Carto 13, 1997, Seattle, WA. 5: ACSM/ASPRS, p. 11-22.

___. Map Algebra Extended with Functors for Temporal Data. In: Perspectives in Conceptual Modeling: ER 2005 Workshops, 2005, Klagenfurt, Austria. Springer, p. 194-207.

Mennis, J.; Viger, R.; Tomlin, D. Cubic Map Algebra Functions for Spatio-Temporal Analysis. **Cartography and Geographic Information Science**, v. 32, n. 1, p. 17-32

OGC. **OpenGIS Simple Features Specification for SQL**. Boston: Open GIS Consortium, 1998.

PEYTON JONES, S. Haskell 98 Language and Libraries the Revised Report 2002.

Peyton Jones, S.; Hughes, J.; Augustsson, L. **Haskell 98: A Non-strict, Purely Functional Language**. 1999. Disponível em: http://www.haskell.org/onlinereport/.

Pullar, D. MapScript: A Map Algebra Programming Language Incorporating Neighborhood Analysis. **GeoInformatica**, v. 5, n. 2, p. 145-163

Takeyama, M.; Couclelis, H. Map Dynamics: Integrating Cellular Automata and GIS through Geo-Algebra. **International Journal of Geographical Information Systems**, v. 11, n. 1, p. 73-91

Thompson, S. **Haskell:The Craft of Functional Programming**. Harlow, England: Pearson Education, 1999.

Tomlin, C. D. **Geographic Information Systems and Cartographic Modeling**. Englewood Cliffs, NJ: Prentice-Hall, 1990.

Winter, S. Topological Relations between Discrete Regions. In: Egenhofer, M.; Herring, J. (Ed.). **Advances in Spatial Databases—4th International Symposium, SSD '95, Portland, ME**. Lecture Notes in Computer Science, v. 951. Berlin: Springer-Verlag, 1995, p. 310-327.

Winter, S.; Frank, A. Topology in Raster and Vector Representation. **GeoInformatica**, v. 4, n. 1, p. 35-65