Center for Spatially Integrated Social Science

Variogram Analysis

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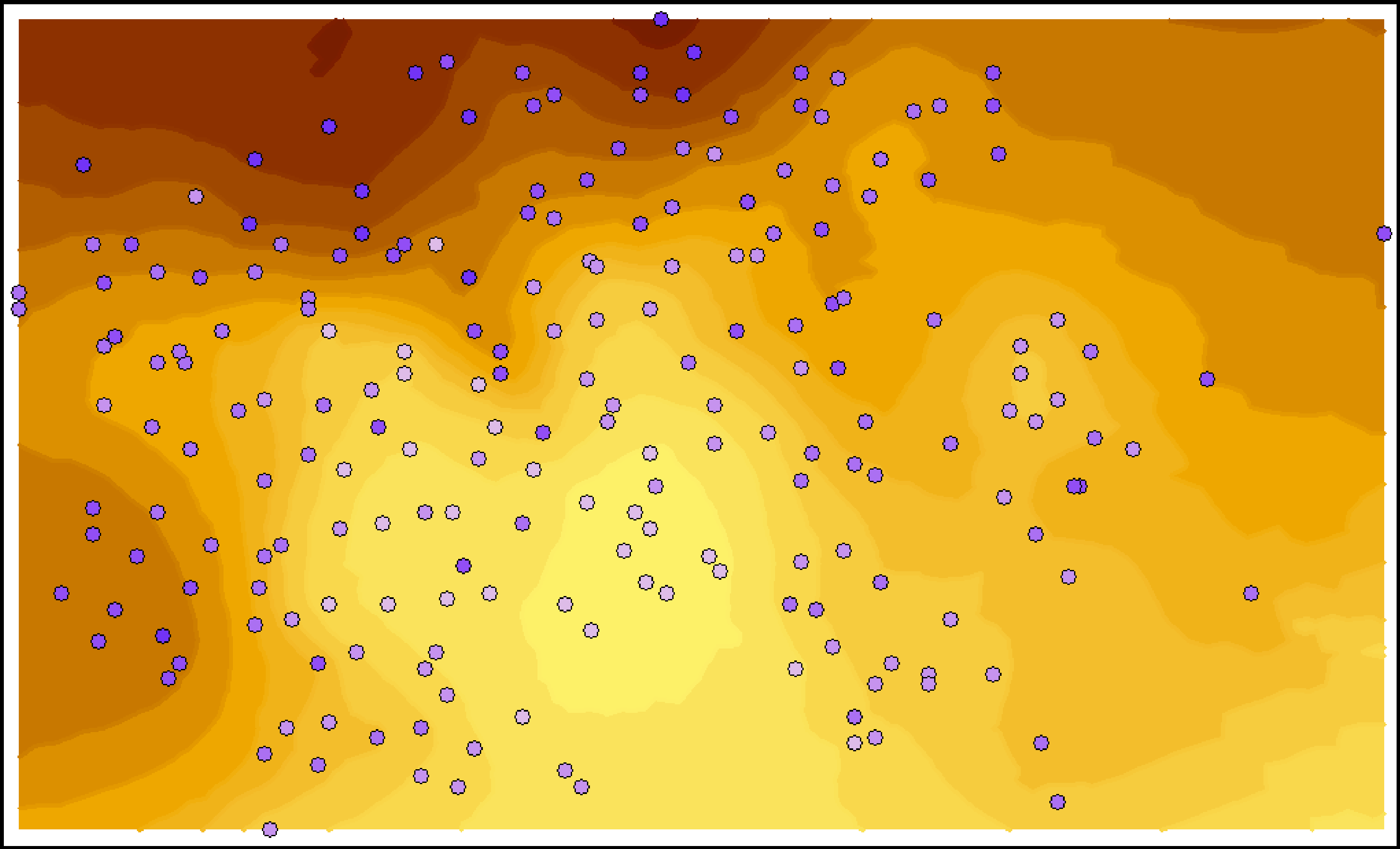
Outline

- Geostatistical Perspective
- Variogram and Correlogram
- Modeling Variograms

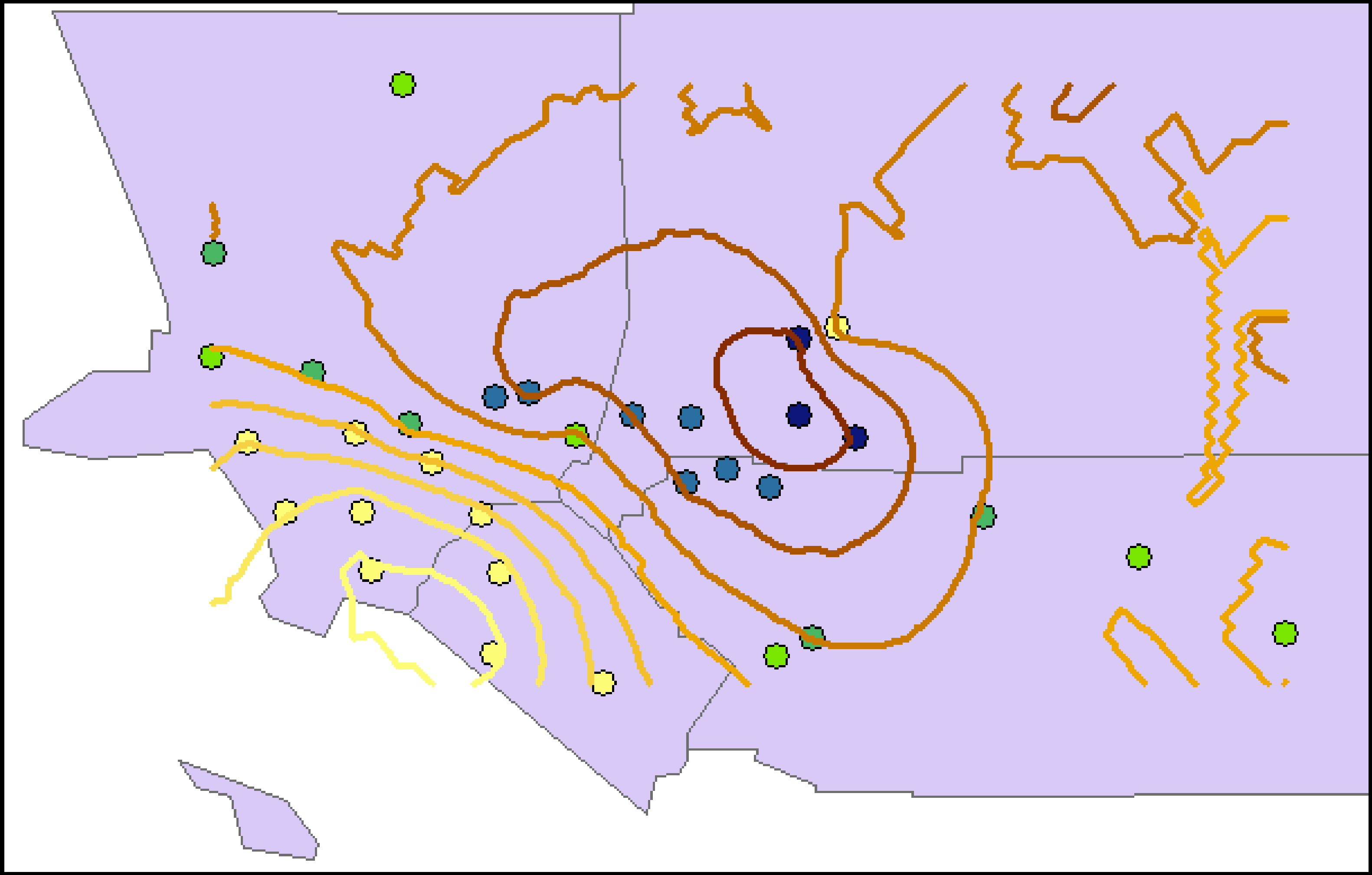
Geostatistical Perspective

Geostatistical Perspective

- Continuous Spatial Index
  - $\{ Z(s): s \in D \}$
    - s is spatial index
    - continuous in R^d
  - sample of spatial locations
    - $\{ s_1, s_2, \ldots, s_N \}$ = locations
    - $\{ Z(s_1), Z(s_2), \ldots, Z(s_N) \}$ = random variable at locations
- Spatial Random Field
  - model for a continuous spatial process
  - field data model



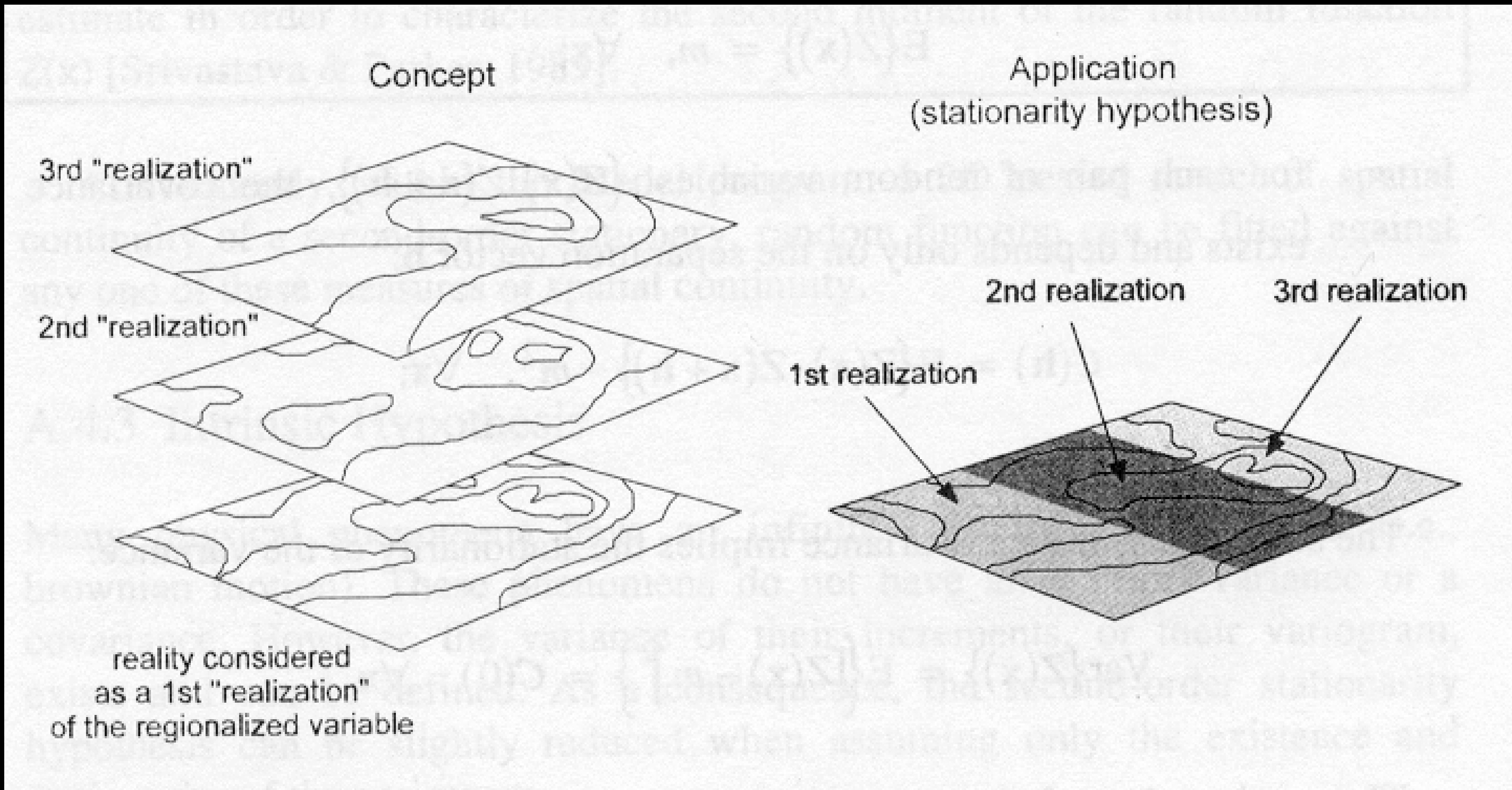
Residential Sales Price, Baltimore MD (1980)
sample points (darker is higher) and contours



Air Quality (Ozone) in Los Angeles Basin
Contours from Spherical Variogram Interpolation

Conceptual Framework

- Equilibrium = Stationarity
  - stochastic process
    - not multiple realizations, but single realization
    - the map is a single data point
  - notion of stability needed to relate sample (= single observation) to population
    - even though there is only one data point, act as if there are multiple observations



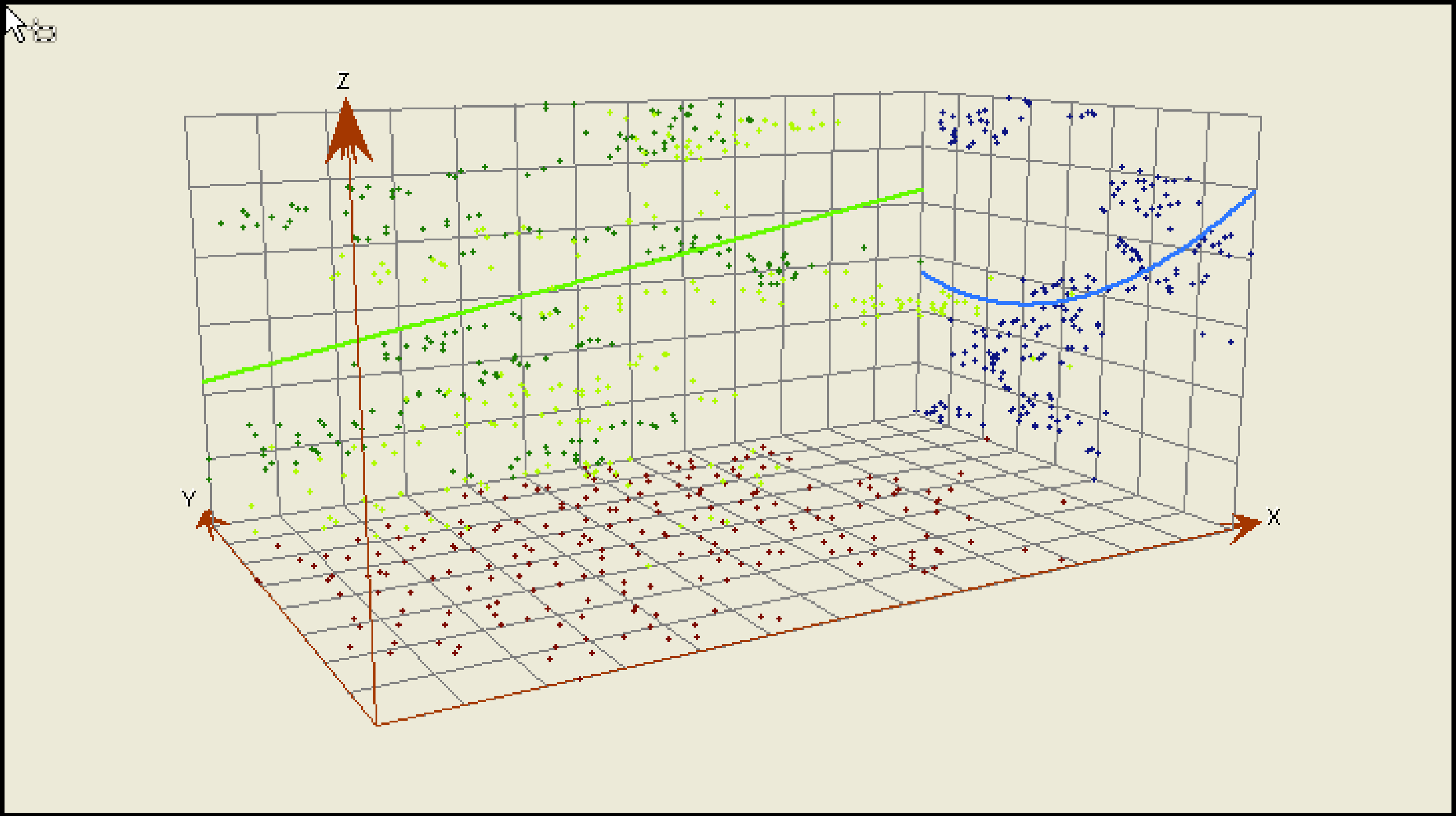
Implications of Spatial Stationarity
(source Y. Pannatier, Variowin)

Moment Conditions

- Constraints on Variability
  - ergodicity
    - average over single realization same as over all possible
    - whether you see one or many maps, information is same
  - moments must exist
    - no infinite variance
  - moments must be “regular” over space
    - restrictions on heterogeneity, on range of dependence

Spatial Stationarity

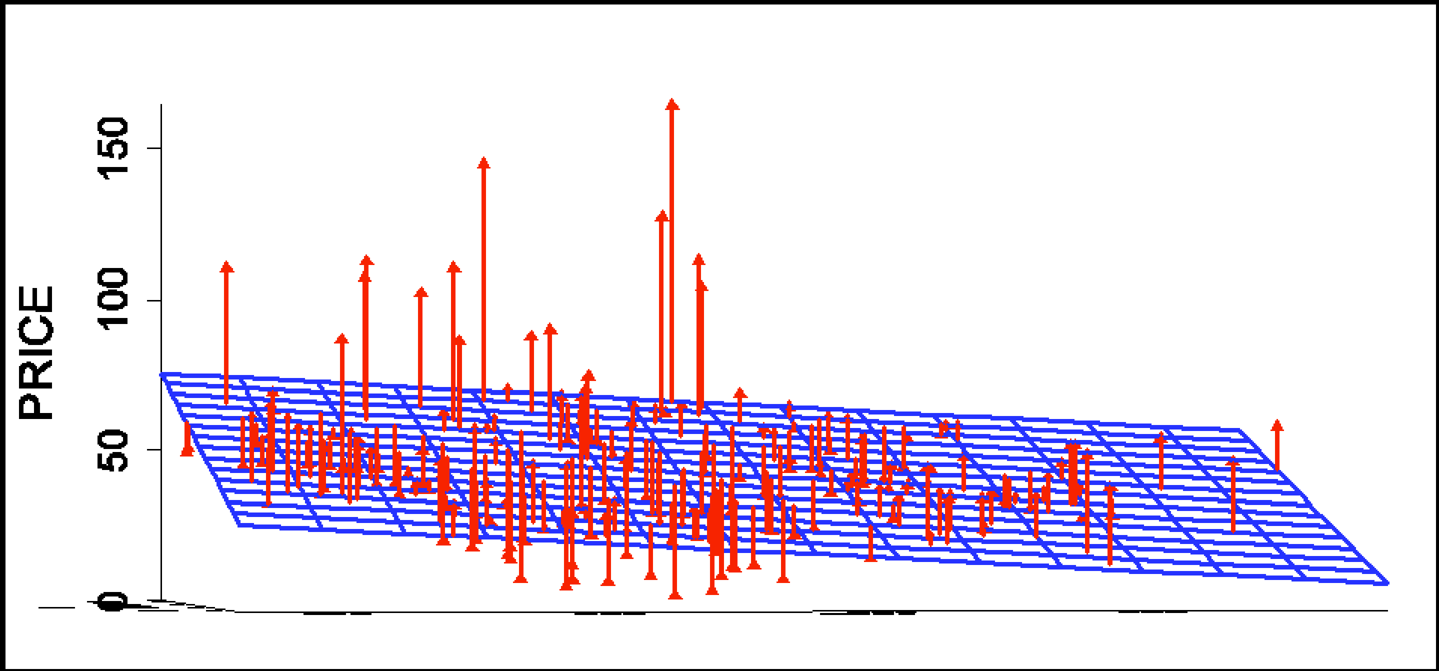
- Strict Stationarity
  - invariance of joint probability density function under spatial shift ("translation")
    - $\{ z(x_1), \ldots , z(x_k) \}$ and $\{ z(x_{1+h}), \ldots , z(x_{k+h}) \}$
    - information about process the same no matter where it is obtained
- Moment Stationarity
  - moments invariant under shift
    - constant mean and constant variance
    - covariance only function of spatial separation h



Baltimore House Sales Prices
Spatial Trend Analysis

Linear Spatial Trend

Baltimore House Prices



$$P = -166.02 - 0.148\ X + 0.634\ Y \qquad R^2 = 0.27$$

Variogram and Correlogram

Variogram

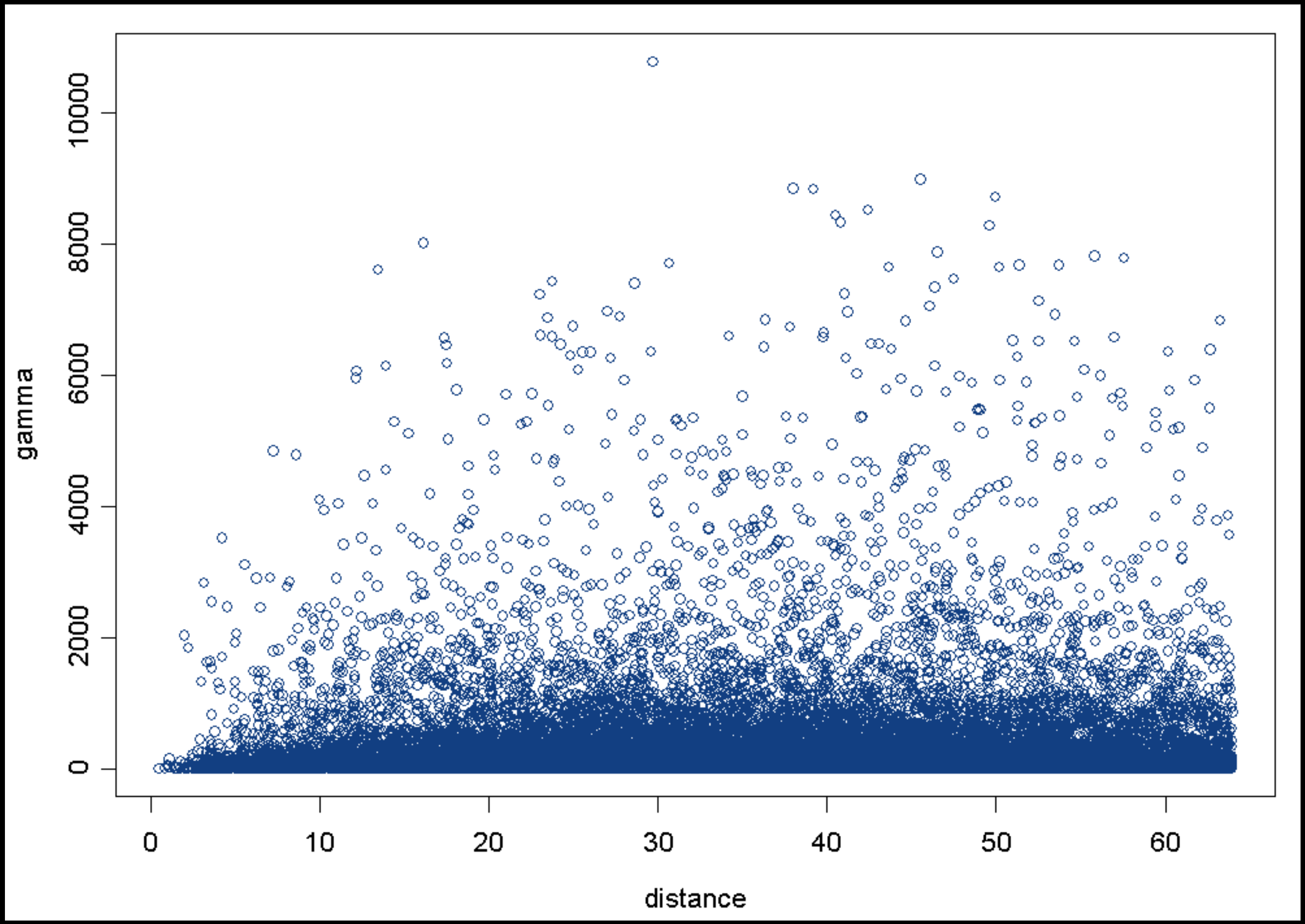
- Intrinsic Hypothesis
  - no spatial trend
    - if there is a trend, take it out
    - residuals have no trend by construction (mean = 0)
  - variance constant
  - variance of first differences only a function of displacement
    - Var { Z[s+h] – Z[s] }
    - how does the variability of the difference change with h

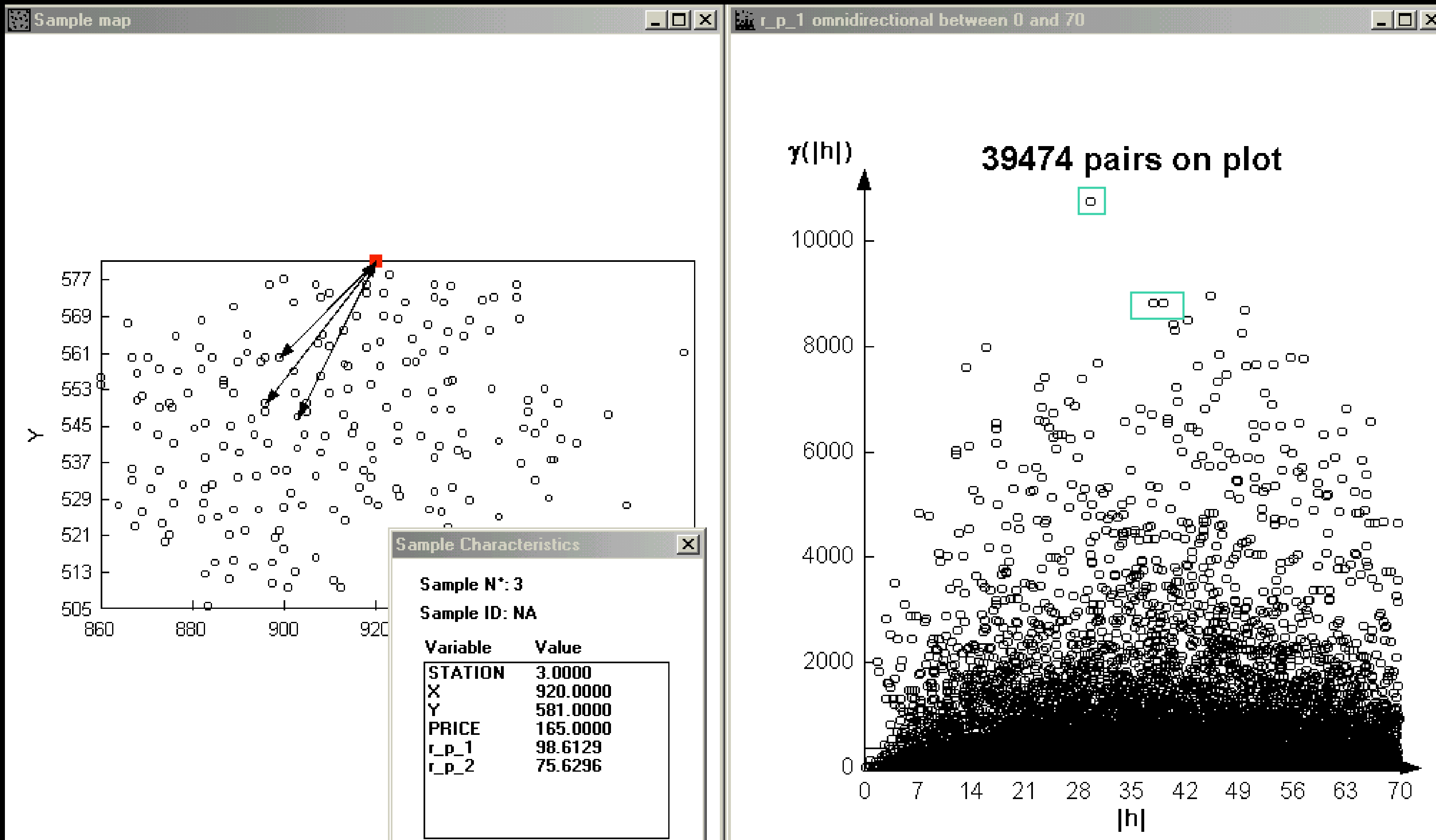
Semi-Variogram

- General Variogram Function
  - $2\gamma(h) = \text{Var}\ [Z(s+h) - Z(s)]$
    - note the factor 2, hence $\gamma(h)$ is 1/2 of variogram or semivariogram
- Regular Case
  - for intrinsic hypothesis, $E[Z(s+h)]=E[Z(s)]$
    - constant mean, hence $E[Z(s+h) - Z(s)] = 0$
    - $\text{Var}[Z(s+h) - Z(s)] = E\{[Z(s+h) - Z(s)] - E[Z(s+h) - Z(s)]\}^2$
  - $2\gamma(h) = E\ [Z(s+h) - Z(s)]^2$
    - average of squared differences

(Semi)Variogram Cloud Plot

(residuals of Baltimore trend surface)



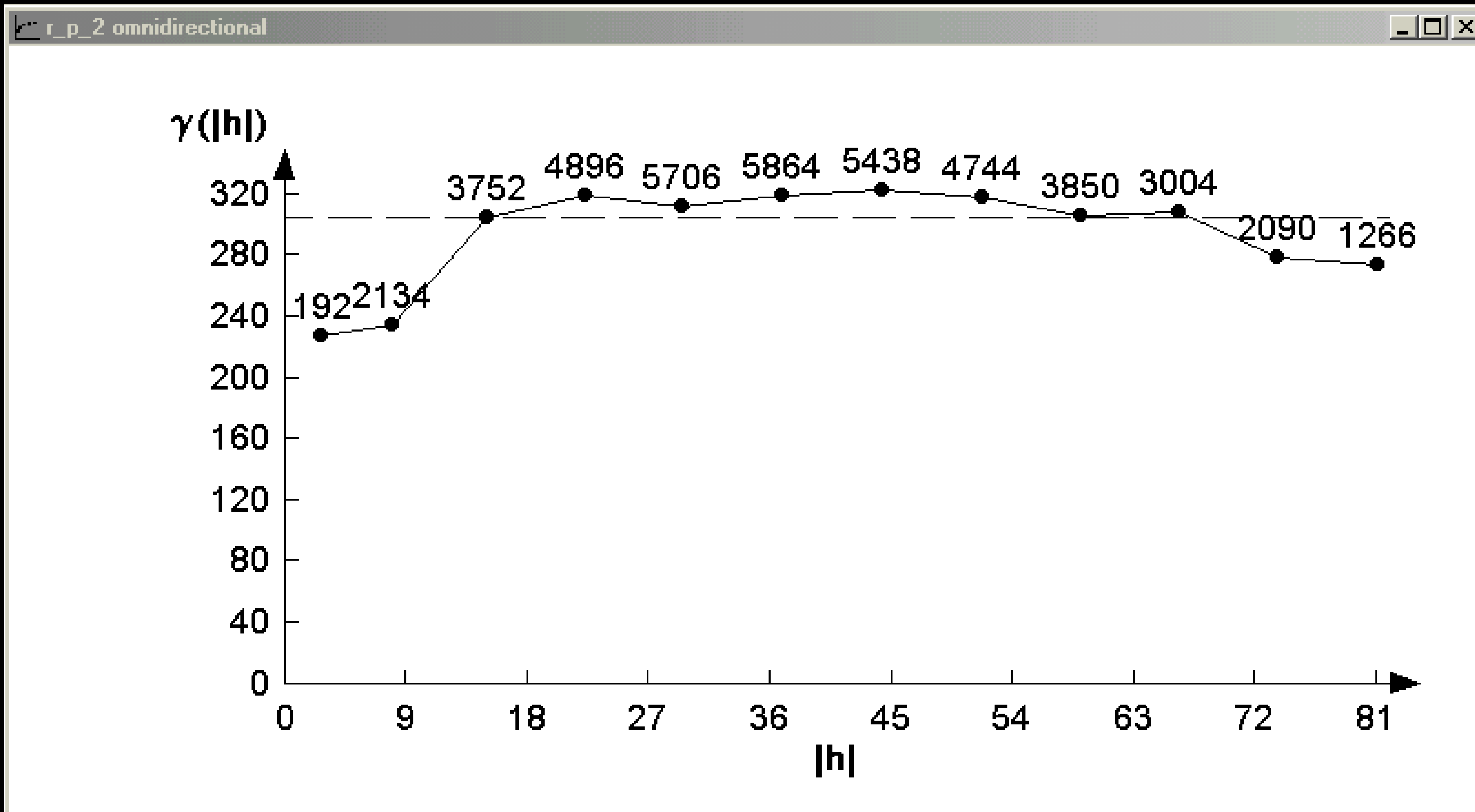


Brushing a Variogram Cloud Plot

identification of potential outliers

Estimating a Variogram

- Method of Moments
- $2\gamma(h) = \{ 1 / | N(h) | \} \times \Sigma_{N(h)} [Z(s+h)-Z(s)]^2$
  - average of squared differences by distance bin
    - h: distance bin
    - N(h): number of pairs in distance bin for h
- Rules of Thumb
  - at least 30 pairs in each bin
  - $h < D / 2$ (D is max distance)
    - distance of reliability



Variogram for residuals from second order trend surface

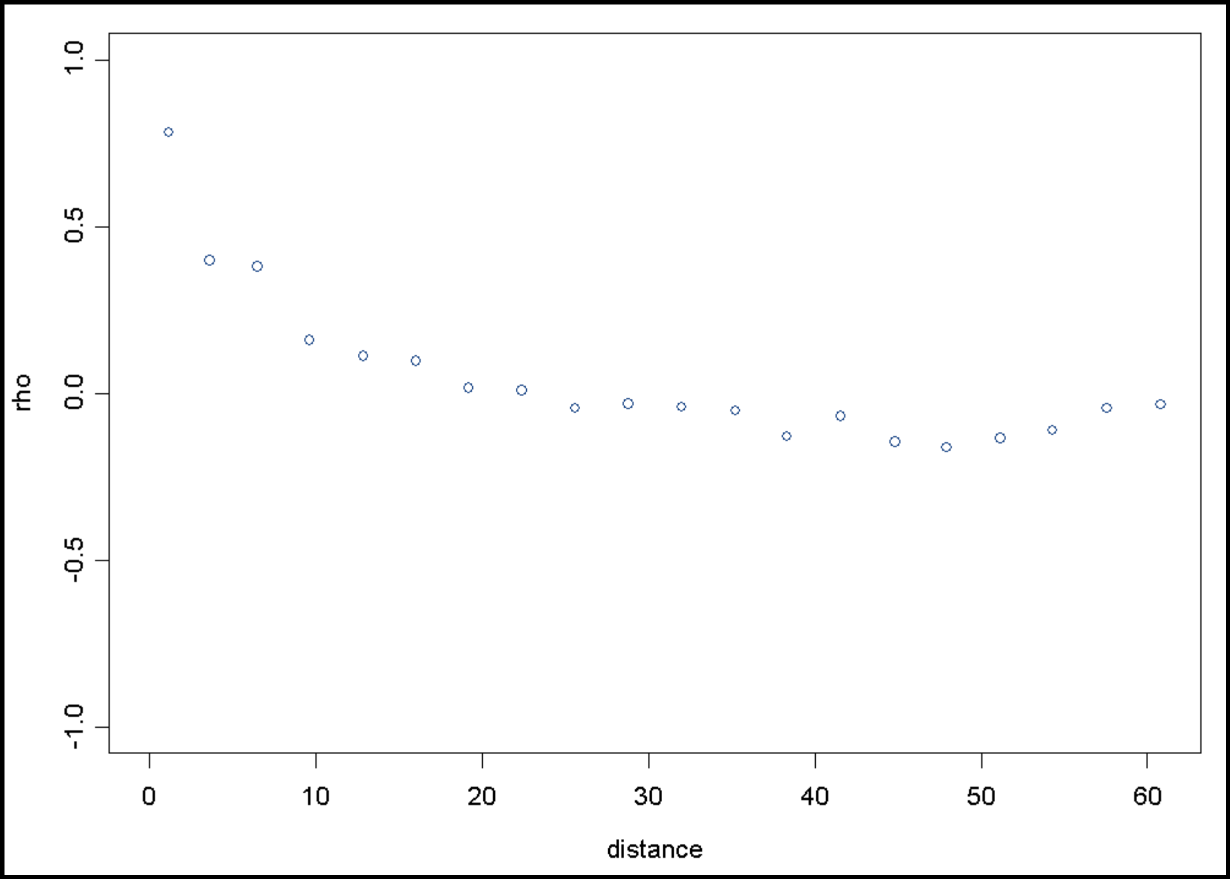
Covariogram

- Second Order Stationarity
  - covariance "regular" over space
  - assumptions stronger than for (semi) variogram
  - define regularity for second order moments
- Covariogram
  - C(h) = Cov [Z(s+h),Z(s)]
    - covariance as a function of "distance" h
  - C(0) = Var [Z(s)]
    - at zero distance, covariance = variance of process

Correlogram

- Correlogram
  - $\rho(h) = C(h) / C(0)$
  - notion of an "autocorrelation" function
  - covariogram C(h) standardized by variance C(0)
- Semi-Variogram and Correlogram
  - $2\gamma(h) = E \{ [Z(s+h) - Z(s)]^2 \}$
    $= 2 E \{Z(s)\}^2 - 2 E \{Z(s+h). Z(s)\}$
  - $\gamma(h) = C(0) - C(h)$
  - correlogram decreases with distance
  - semivariogram increases with distance

Correlogram
(Baltimore trend surface residuals)



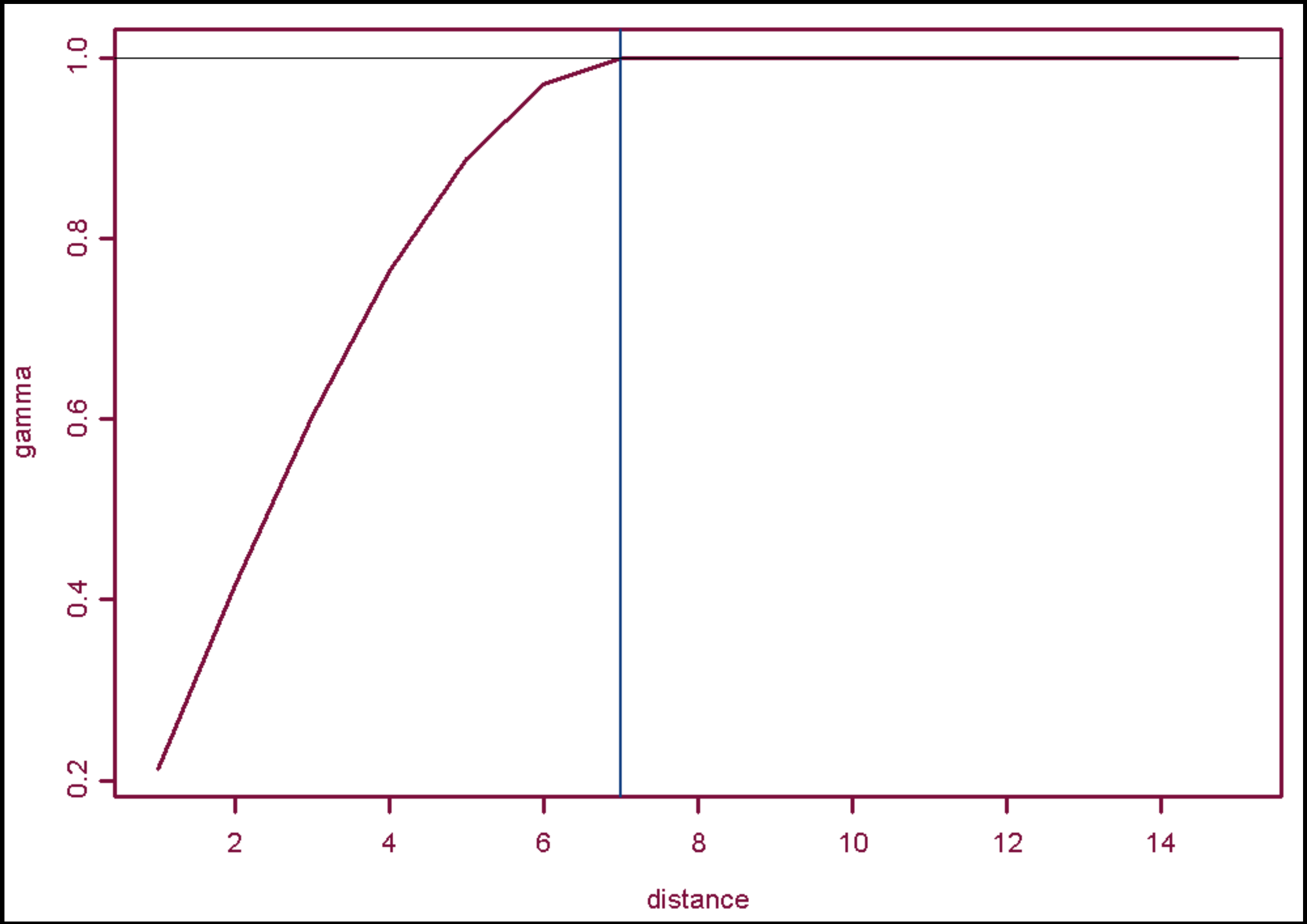
Modeling Variograms

Range and Sill

- Limit Behavior
  - $\gamma(h) = C(0) - C(h)$
  - as $h \rightarrow \infty$, $C(h) \rightarrow 0$ or $\gamma(h) \rightarrow C(0)$
    - limit on range of dependence
- Sill
  - C(0) is sill = Var[Z(s)]
- Range
  - h s.t. $\gamma(h) = C(0)$ is range
    - use range in spatial sampling
    - range is distance beyond which there is no spatial autocorrelation

Theoretical Spherical Variogram

sill = 1



range = 7

Nugget Effect

- Behavior Near $h = 0$
  - $\gamma(h) \rightarrow c_0 > 0$ as $h \rightarrow 0$
  - not possible mathematically
    - $\gamma(0) = 0$ by definition
- Interpretation of Nugget Effect
  - measurement error for $h < h_{min}$
  - scale problems

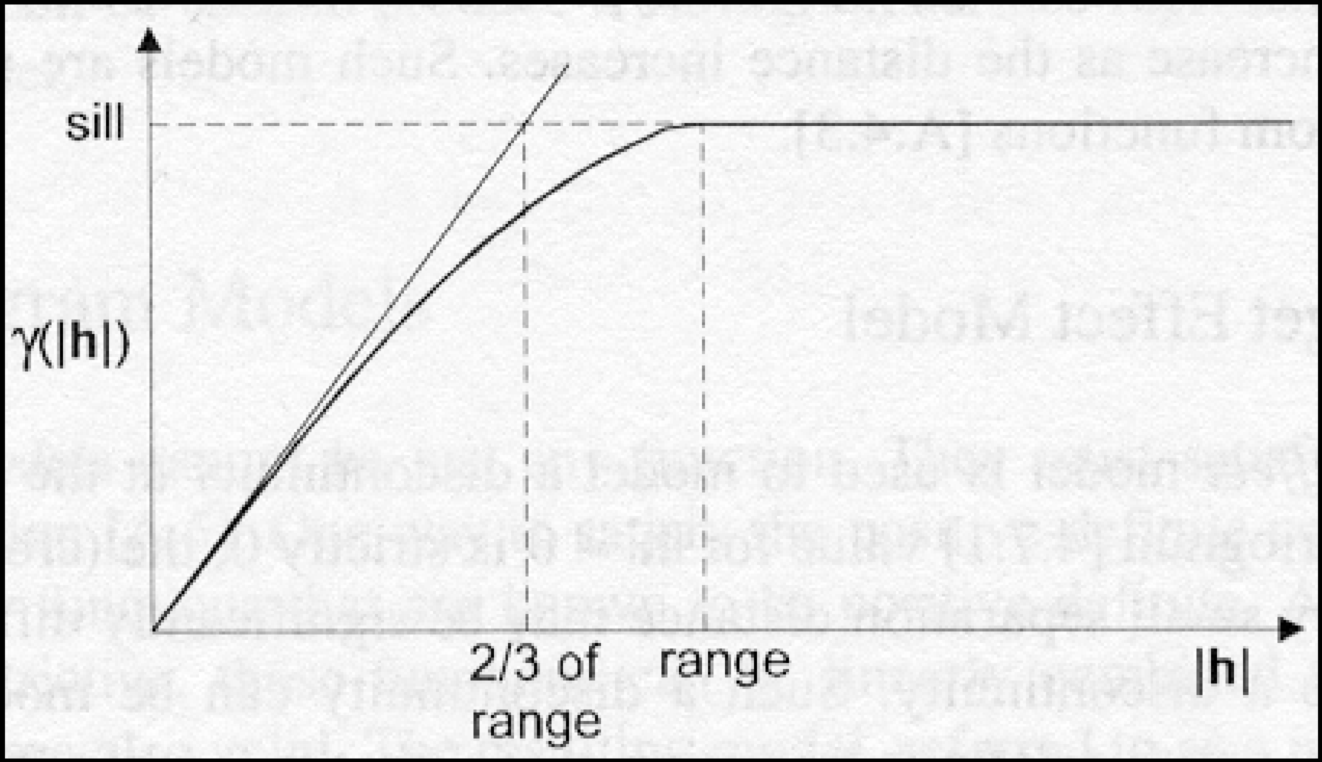
Valid Variogram Models

- Moment Conditions
  - C(•) positive definite ∀ h
  - γ(•) negative definite ∀ h
- Isotropic Variogram Models
  - γ (h, θ) = 0 for h = 0
  - γ (h, θ) same in all directions
- Non-Isotropic Models
  - directional
    - different variogram for different directions

Examples of Variogram Models

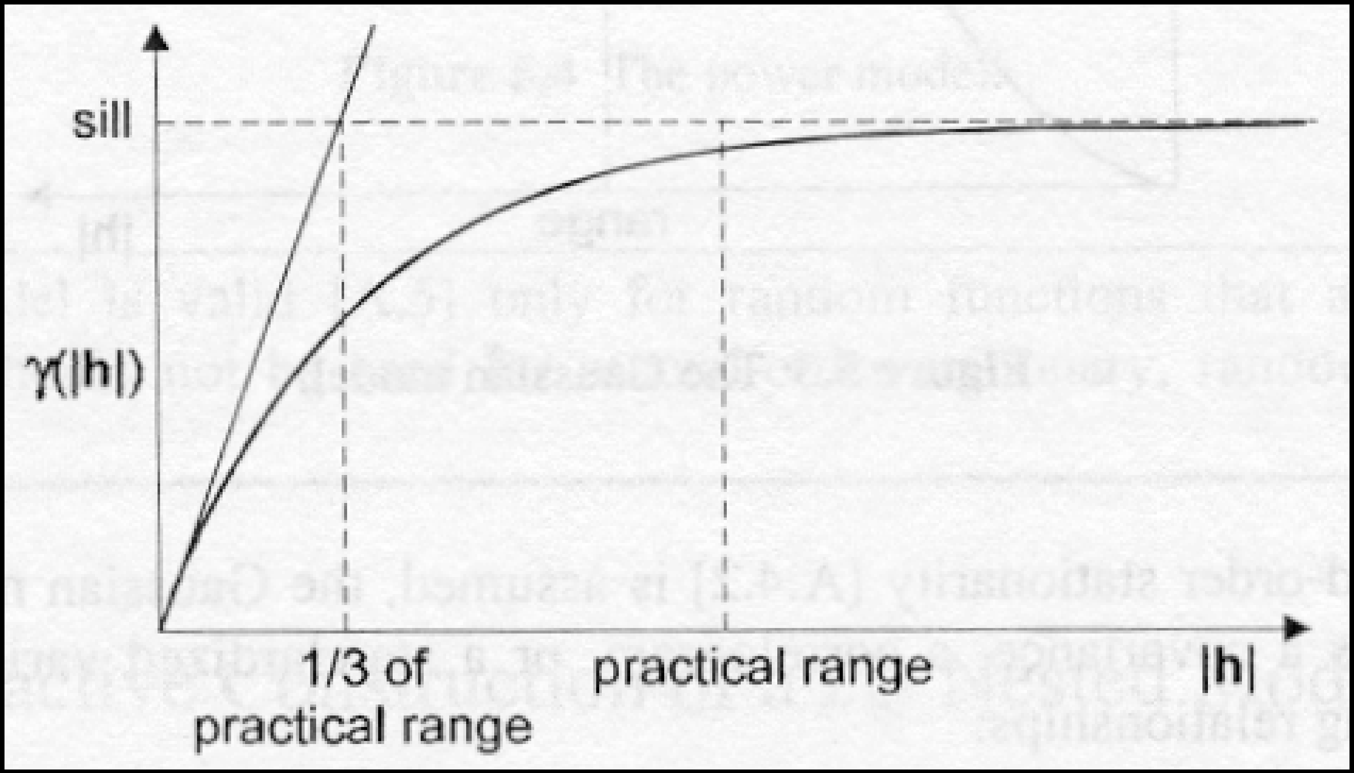
- Spherical
  - $\gamma(h, \theta) = c_0 + c_s\{1.5h/a - 0.5(h/a)^3\}$ for $0 < h \leq a$
    $= c_0 + c_s$ for $h \geq a$
  - c_0 = nugget effect, $c_0 + c_s$ = sill, a = range
- Exponential
  - $\gamma(h, \theta) = c_0 + c_s\{ 1 - e^{-(3h/a)} \}$
  - a is "practical range" 95% of asymptotic range

Spherical Variogram



Source Pannatier 96

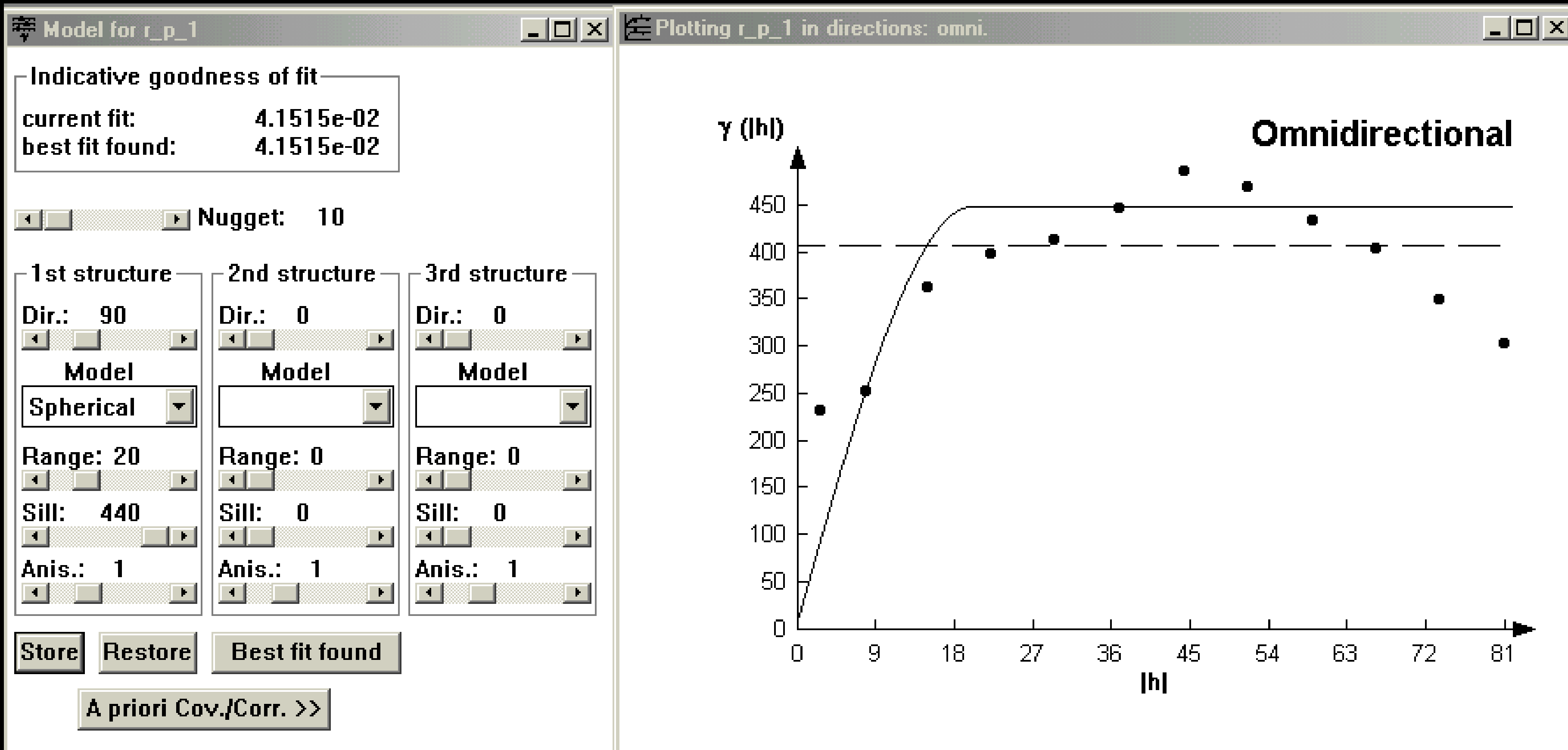
Exponential Variogram



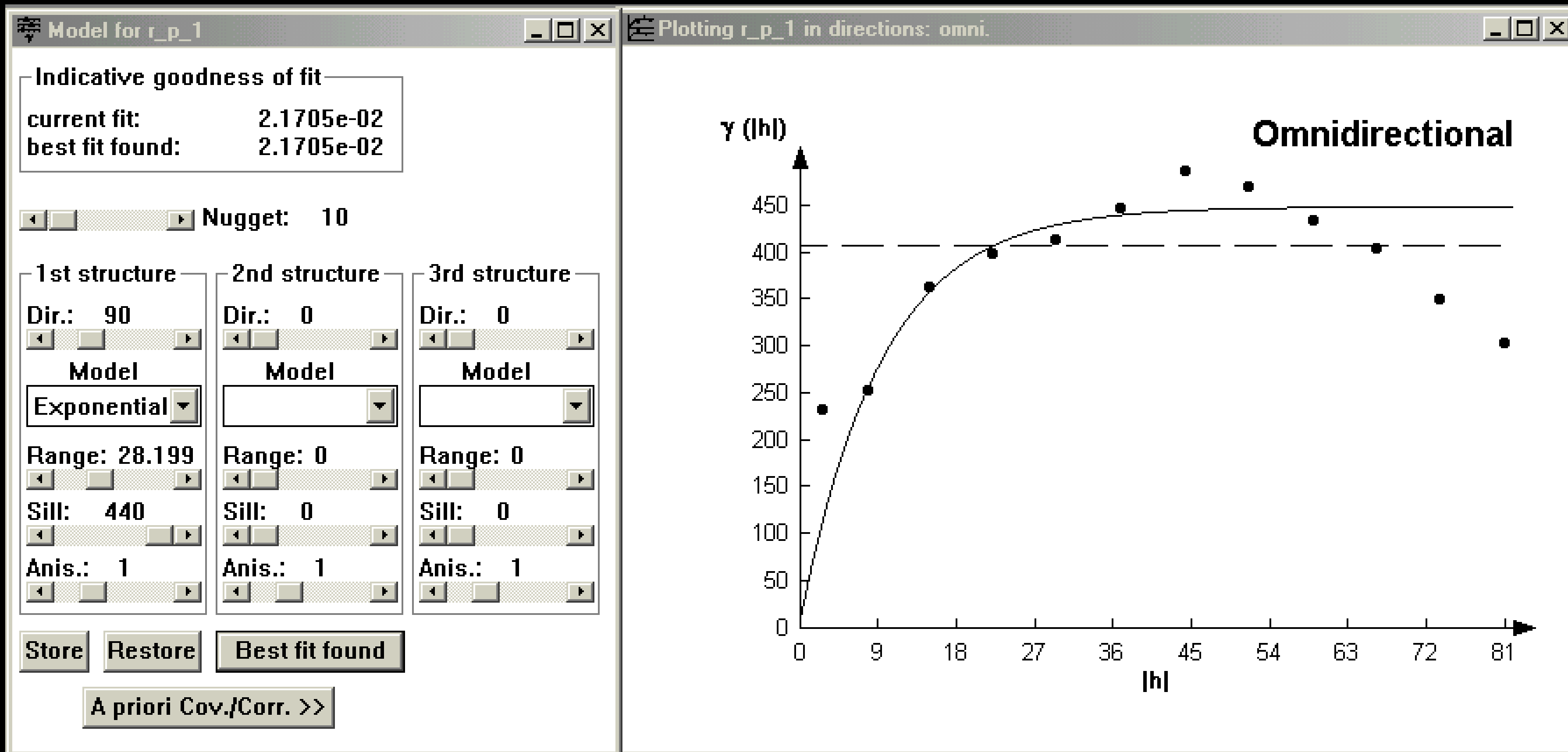
Source Pannatier 96

Fitting a Variogram Model

- Parameter Estimates
  - from empirical variogram
  - methods
    - nonlinear least squares
    - weighted least squares
    - "eyeball"
- Parameters must satisfy constraints
  - valid parameter space



Fit of Spherical Variogram Model to Baltimore Residuals



Fit of Exponential Variogram Model to Baltimore Residuals

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