Spatial Autocorrelation (2)
Spatial Weights

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Outline

- Concepts
- Contiguity Weights
- Other Spatial Weights
- Spatial Lag Operator
Concepts
Why Spatial Weights?

- **Spatial Correlation**
  - $\text{Cov}[y_i, y_h] \neq 0$, for $i \neq h$

- **Structure of Correlation**
  - which $i, h$ interact?
  - $N$ observations to estimate $N(N-1)/2$ interactions
  - impose structure in terms of what are the “neighbors” for each location
Example: N=6

contiguity = common boundary
contiguity as a graph
link between nodes = contiguity
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First Order Contiguity Matrix
Spatial Weights Matrix

- **Definition**
  - N by N positive matrix $W$, with elements $w_{ij}$

- **Simplest Form: Binary Contiguity**
  - $w_{ij} = 1$ for i and j “neighbors” (e.g. $d_{ij} < \text{critical distance}$)
  - $w_{ij} = 0$ otherwise, $w_{ii} = 0$ by convention

- **Row Standardization**
  - averaging of neighboring values
  - $w_{ij}^s = w_{ij} / \sum_j w_{ij}$ such that $\sum_j w_{ij}^s = 1$
  - spatial parameters comparable
How to Construct Weights

- Contiguity
  - common boundary

- Distance
  - distance band
  - k-nearest neighbors

- General
  - social distance
  - complex distance decay functions
Contiguity Weights
Contiguity – Regular Grid

- **Regular Grid**
  - rook
    - 2, 4, 6, 8
  - bishop
    - 1, 3, 7, 9
  - queen
    - both

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Regular Grid Polygons
Contiguity and GIS Precision

- The regular “grid” in the top picture is actually misaligned at the corners.
- The grid on the bottom picture has the proper contiguity.
Contiguity – Irregular Units

- Irregular Units
  - common border
    - rook
  - common vertex
    - 039 and 067
    - queen
Contiguity Weights in GeoDa
Higher Order Contiguity

- **Recursive Definition**
  - $j$ contiguous to $i$ of order $p$:
    - $j$ first order contiguous to $k$
    - $k$ contiguous to $i$ of order $p-1$
  - $i$ and $j$ not already contiguous of lower order

- **Spatial Weights Matrix**
  - representation of network or graph
contiguity as a graph
link between nodes = contiguity
Circularity and Redundancy in Higher Order Weights

- Powering of the Weights Matrix
  - standard approach invalid for contiguity
- Removing Circularity and Redundancy
  - sparse network representation of weights
  - modified Dijkstra algorithm to identify number of steps between nearest neighbors (Anselin and Smirnov)
  - number of steps = order of contiguity
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### Matrix Constructed by Bottom-Up Algorithm

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Other Spatial Weights
Distance-Based Weights

- Distance Band
  - \( w_{ij} = 1 \) for \( d_{ij} < \) distance cut-off

- K-Nearest Neighbor Weights
  - \( k \) neighbors, irrespective of actual distance
  - “warps” space
Distance Based Weights

Distance Weight
- Threshold Distance: 0.824621

Distance Weight
- Threshold Distance: 1
- k-Nearest Neighbors: 3

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General Spatial Weights

- Cliff-Ord Weights
  - \( w_{ij} \) to reflect potential spatial interaction between \( i \) and \( j \)
  - \( w_{ij} = [d_{ij}]^{-a} \cdot [b_{ij}]^b \)
    - with
      - \( d_{ij} \) as distance between \( i \) and \( j \)
      - \( b_{ij} \) as share of common boundary between \( i \) and \( j \) in perimeter of \( i \)
General Spatial Weights
(continued)

- Weights May Contain Parameters
  - inverse distance weights
    - \( w_{ij} = \frac{1}{d_{ij}^{\alpha}} \)
  - estimated from data or chosen a priori
    - in practice: second power (gravity model)
  - identification problems in nonlinear weights
    - interaction is multiplicative: \( \rho \cdot w_{ij} = \rho \left( \frac{1}{d_{ij}^{\alpha}} \right) \)
    - parameters \( \rho \) and \( \alpha \) not separately identified
General Spatial Weights
(continued)

- Economic Weights (Case)
  - block structure, state effect
    - $w_{ij} = 1$ for all $i, j$ in "block"
  - economic distance $|r_i - r_j|$
  - economic weight $= 1/|r_i - r_j|$
    - e.g., $r =$ total employment
Characteristics of Spatial Weights

- Measures of Overall Connectedness
  - percent nonzero weights (sparseness)
  - average weight
  - average number of links
  - principal eigenvalue

- Location-Specific Measures
  - most/least connected observations
  - unconnected observations = islands
NC Counties with 4 Neighbors
Spatial Lag Operator
Spatial Shift

- No Direct Counterpart to Time Series Shift Operator
  - time series: $L_k = y_{t-k}$
  - spatial series: which $h$ are shifted by “$k$” from location $i$?
    - on regular lattice: east, west, north, south
    - $(i - 1, j)$ $(i + 1, j)$ $(i, j - 1)$ $(i, j + 1)$
  - arbitrary for irregular lattice
    - different number of neighbors by observation
Spatial Lag Operator

- Distributed Lag
  - row-standardized weights $\Sigma_j w_{ij} = 1$
  - spatial lag is weighted average of neighboring values
    - $\Sigma_j w_{ij}y_j$, for each i
  - vector $Wy$
  - spatial lag does not contain $y_i$
- spatial lag is a smoother
  - not a window average
value $y_i = \$42,300$

4 neighbors
values for neighbors: $\$50,200, \$64,600, \$45,000, \$34,200$

spatial lag = $(1/4)\$50,200 + (1/4)\$64,600 + (1/4)\$45,000 + (1/4)\$34,200 = \$48,500$
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Map Legend

Green areas represent the counties listed in the table.
### Spatial Lags in GeoDa Table

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